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Division of Electromagnetic Research

RESEARCH REPORT No. EM-191

On the Principle of Limiting Absorption

D. M. EIDUS (Leningrad)

TRANSLATED FROM THE RUSSIAN

By CLYDE D. HILL

AUGUST, 1963

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FOREWORD

In many practical problems of electromagnetic theory the sources producing the electromagnetic field have a sinusoidal time dependence. It is then assumed that the resulting electromagnetic field also has this time dependence and the resulting problem then becomes a simpler problem which is essentially time independent.

It has been long assumed that this method of analysis yields the steady state solution. However, it was not until recent time that attempts were made to prove this fact. Because of the continuing interest in this problem the Division of Electromagnetic Research felt it would be worthwhile to publish this translation of a paper by the Soviet scientist D. M. Eidus. This paper by Eidus treats the problem in the greatest generality which has yet been undertaken. Since he considers a very general equation and boundary his results contain and extend the results of other workers in this field.

A parallel problem also considered by Eidus in this paper concerns the principle of limiting absorption. In practical problems this principle states that if a boundary value problem is solved for a medium which is slightly conducting and then the conductivity is allowed to approach zero the resulting limit of the solution is the solution of the same boundary value problem in a non-conducting medium. This principle is important since it is often simpler in the mathematical analysis to introduce an artificial conductivity in order that a solution may be more readily analyzed. Again the work of Eidus on this problem is the most general available.

Because of the widespread interest in this paper the translation was performed and reproduced as a service to interested scientists.

ON THE PRINCIPLE OF LIMITING ABSORPTION*

D. M. EIDUS (LENINGRAD)

We shall consider an elliptic equation of the form

$$gu \equiv - \sum_{k,\ell=1}^{m} \frac{\partial}{\partial x_k} \left(a_{k\ell}(x) \frac{\partial u}{\partial x_\ell} \right) + q(x)u = \lambda u + f(x) \tag{1}$$

in an infinite region Ω of the m-dimensional space R_m subject to the boundary condition

$$u|\Gamma = 0 \quad . \tag{2}$$

Here Γ is the boundary of the region Ω and λ is a point in the continuous spectrum of our operator, being considered in the complex space $L_2(\Omega)$. For the boundary value problem formulated as such, uniqueness of the solution does not hold. Moreover the existing solutions do not in general belong to $L_2(\Omega)$. The principle of limiting absorption consists in the following: The solution of the boundary value problem is defined as the limit as $\varepsilon \longrightarrow 0$ (or as $\varepsilon \longrightarrow -0$) of solution of the equation

$$gu = (\lambda+\varepsilon i)u + f \quad , \tag{3}$$

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which satisfy condition (2) and belong to $L_2(\Omega)$.

This principle requires justification. Namely, it should be proved that the limit mentioned exists and is a solution of the boundary value problem (1), (2). In [1] and [2] A. G. Sveshnikov carried out this demonstration for the equation $-\Delta u = \lambda u + f$ in the layer between two parallel planes and in an infinite cylinder. In [3] A. Y. Povzner considered the case of the equation $-\Delta u + qu = \lambda u + f$ in the whole space R_3. The function $q(x)$ was assumed to be continuous and decreasing at infinity like $|x|^{-(3+\alpha)}$, where $\alpha > \frac{1}{2}$ (here and in what follows, $|x|$ denotes the distance of the point x from the origin).

The present work is concerned with the case of a region with a finite boundary and, in addition, with the case where the infinite part of the boundary is cylindrical. Beside the principle of limiting absorption, we shall also consider the well-known principle of limiting amplitude. The basic results of the present work were formulated in the note [4].

Introduction

We shall assume that the functions $a_{k\ell}(x)$, where $a_{k\ell} = a_{\ell k}$, are real and continuously differentiable in $\Omega + \Gamma$. Moreover for any vector $\xi = \xi_1, \xi_2, \dots, \xi_m$ and any point $x \in \Omega$ we shall assume the inequalities[1]

$$\sum_{k,\ell=1}^{m} a_{k\ell}\, \xi_k\, \overline{\xi}_\ell \geq c_1 |\xi|^2 \quad , \tag{4}$$

$$\sum_{k,\ell=1}^{m} a_{k\ell}\, \xi_k\, \overline{\xi}_\ell \leq M\,(|x|)\,|\xi|^2 \quad , \tag{5}$$

are satisfied. Here $M(r)$ is some positive function, defined for $r > 0$, for which[2]

$$\int_1^\infty [M(r)]^{-\frac{1}{2}}\, dr = \infty \quad . \tag{6}$$

We shall assume the function $q(x)$ is real, measurable and bounded in any finite subregion, and in Ω also satisfies the inequality

$$q(x) \geq c_2 \quad . \tag{7}$$

The function $f(x)$ we will consider to belong to $L_2(\Omega)$.

We put

$$g_0 u = gu - qu \quad ,$$

$$\tilde{g}u = g_0 u + \tilde{q}u \quad ,$$

where $\tilde{q}(x) = q(x) + |c_2| + 1$ (then $\tilde{q}(x) \geq 1$). Let J be the totality of all functions $u \in W_2^{(1)}(\Omega)$, for which

$$E(u,u) \equiv \int_\Omega \left[\sum_{k,\ell=1}^{m} a_{k\ell} \frac{\partial u}{\partial x_k} \frac{\partial \overline{u}}{\partial x_\ell} + \tilde{q} u \overline{u} \right] dx < \infty \quad .$$

We introduce on J the scalar product $E(u,v)$. Then J is a complete Hilbert space. Note that if the functions $a_{k\ell}$ and q are bounded in Ω, then the norms in the spaces J and $W_2^{(1)}(\Omega)$ are equivalent.

We shall denote by $\overline{W}_2^{(k)}(\omega)$ the class of functions which are defined in the region ω and belong to $W_2^{(k)}(\omega')$ for any compact subset ω' of ω. We will make use of the following well-known inequality (see [6]), holding for functions $\phi \varepsilon W_2^{(2)}(\omega")$:

$$||\phi||W_2^{(2)}(\omega') \leq c_3 \left[||\phi||_{L_2(\omega")} + ||g_o\phi||_{L_2(\omega")} \right] \tag{8}$$

Here ω' is a finite interior subregion of the region $\omega"$ and the constant c_3 depends on ω' and $\omega"$.

Let the region ω be bounded. As usual we introduce the class $\overset{o}{D}(\omega)$, obtained by completing under the metric of $W_2^{(1)}(\omega)$ the set of functions which are defined in ω and each of which vanishes in some boundary strip. We denote the sphere of radius ρ centered about the origin by S_ρ. Let Ω_ρ be the part of Ω contained in S_ρ, and σ_ρ be the part of S_ρ belonging to $\Omega + \Gamma$. We put $\overset{o}{D}(\Omega_\rho) = \overset{o}{D}_\rho$ and introduce, in addition, the class $\overset{o}{D}{}'_\rho$ as the completion under the metric of $W_2^{(1)}(\Omega_\rho)$ the set of functions which are defined in Ω_ρ and each of which vanish in some strip along Γ_ρ (the strips in question depend on the function).

We set $\Phi_\rho = \overset{\circ}{D}_\rho \cap \overline{W}_2^{(2)}(\Omega_\rho)$, $\Phi_\rho' = \overset{\circ}{D}_\rho' \cap \overline{W}_2^{(2)}(\Omega_\rho)$.

LEMMA 1. Let $u \in \Phi_{\rho+2}$, $g_o u \in L_2(\Omega_{\rho+2})$. Then

$$||u||_{W_2^{(1)}(\Omega_\rho)} \leq c_4(\rho)\left[||u||_{L_2(\Omega_{\rho+2})} + ||g_o u||_{L_2(\Omega_{\rho+2})}\right] \tag{9}$$

Proof: We introduce the function $b_\rho(r)$, where $r = |x|$, in the following manner:

$$b_\rho(r) = 1 \text{ for } r < \rho$$

$$b_\rho(r) = \sin^2(\rho-r+1)\frac{\pi}{2} \text{ for } \rho \leq r \leq \rho + 1$$

$$b_\rho(r) = 0 \text{ for } \rho + 1 \leq r \leq \rho + 2.$$

We put $v = b_\rho u$, then $v \in \Phi_{\rho+2}$. We have:

$$\int_{\Omega_{\rho+2}} v g_o \overline{u} dx = \int_{\Omega_{\rho+2}} \sum_{k,\ell=1}^{m} a_{k\ell} \frac{\partial v}{\partial x_k} \frac{\partial \overline{u}}{\partial x_\ell} dx,$$

$$\int_{\Omega_{\rho+2}} b_\rho \sum_{k,\ell=1}^{m} a_{k\ell} \frac{\partial u}{\partial x_k} \frac{\partial \overline{u}}{\partial x_\ell} dx = - \int_{\Omega_{\rho+2}} u \sum_{k,\ell=1}^{m} a_{k\ell} \frac{\partial b_\rho}{\partial x_k} \frac{\partial u}{\partial x_\ell} dx$$

$$+ \int_{\Omega_{\rho+2}} b_\rho u g_o \overline{u} dx$$

$$\leq \left(\frac{1}{2} + \frac{1}{2\delta}\right) \int_{\Omega_{\rho+2}} |u|^2 dx + \frac{1}{2} \int_{\Omega_{\rho+2}} |g_o u|^2 dx$$

$$+ \frac{\delta}{2} \int_{\Omega_{\rho+2}} \left[\sum_{k,\ell=1}^{m} a_{k\ell} \frac{\partial b_\rho}{\partial x_k} \frac{\partial \overline{u}}{\partial x_\ell}\right]^2 dx,$$

where δ is any positive number. Making use of the inequality

$$\mathrm{grad}^2 b_\rho \leq \pi b_\rho ,$$

it follows that

$$\left| \sum_{k,\ell=1}^{m} a_{k\ell} \frac{\partial b_\rho}{\partial x_k} \frac{\partial \bar{u}}{\partial x_\ell} \right|^2 \leq c_5(\rho) b_\rho \sum_{k,\ell=1}^{m} a_{k\ell} \frac{\partial u}{\partial x_k} \frac{\partial \bar{u}}{\partial x_\ell} .$$

Then

$$\left(1 - \frac{c_5 \delta}{2}\right) \int_{\Omega_{\rho+2}} b_\rho \sum_{k,\ell=1}^{m} a_{k\ell} \frac{\partial u}{\partial x_k} \frac{\partial \bar{u}}{\partial x_\ell}\, dx$$

$$\leq \frac{1+\delta}{2\delta} \int_{\Omega_{\rho+2}} |u|^2 dx + \frac{1}{2} \int_{\Omega_{\rho+2}} |g_o u|^2 dx \quad .$$

Setting $\delta = c_5^{-1}$, we obtain:

$$\int_{\Omega_\rho} \sum_{k=1}^{m} \left| \frac{\partial u}{\partial x_k} \right|^2 dx \leq c_4(\rho) \int_{\Omega_{\rho+2}} \left| |u|^2 + |g_o u|^2 \right| dx \; .$$

Q.E.D.

We now define the class $\overset{o}{D}$ in the following manner: We shall consider a function Φ to belong to $\overset{o}{D}$ if it is defined in Ω and if there exists a sequence of functions $\Phi_n \in \overline{W}_2^{(1)}(\Omega)$, vanishing in a strip along Γ, such that, in any region Ω_ρ, $\Phi_n \longrightarrow \Phi$ in $W_2^{(1)}(\Omega_\rho)$.

We put $\Phi = \overset{\circ}{D} \cap \overline{W}_2^{(2)}(\Omega)$. We will say that *the sequence* $\phi_n \in \Phi$ *converges in the sense* (Φ) *to the function* ϕ, *if* $\phi_n \longrightarrow \phi$ is any finite interior subregion ω' in the metric of $W_2^{(2)}(\omega')$ and if, in addition, $\phi_n \longrightarrow \phi$ in any region Ω_ρ in the metric $W_2^{(1)}(\Omega_\rho)$ (obviously, then $\phi \in \Phi$).

We define on the functions $\phi \in \Phi$ the differential operator $\overline{G}$ with the aid of the equation $\overline{G}\phi = g\phi$ and replace the boundary value problem (1), (2) by the equation

$$\overline{G}u = \lambda u + f . \tag{10}$$

We introduce the class P in the following manner: $u \in P$ if $u \in J \cap \overline{W}_2^{(2)}(\Omega)$ and $gu \in L_2(\Omega)$. Let $D = \overset{\circ}{D} \cap J$, and (ϕ,ψ) denote the scalar product in $L_2(\Omega)$. We introduce the notation:

$$E_\rho(u,v) = \int_{\Omega_\rho} \left| \sum_{k,\ell=1}^{m} a_{k\ell} \frac{\partial u}{\partial x_k}\frac{\partial v}{\partial x_\ell} + \tilde{q}u\bar{v} \right| dx.$$

LEMMA 2. *Let* $u \in P$, $v \in D$. *Then*

$$(\tilde{g}u,v) = E(u,v).$$

Proof: Since $u \in W_2^{(1)}(\Omega)$, then for almost all $\rho > 0$ the function $\sum_{k=1}^{m} \left| \frac{\partial u}{\partial x_k} \right|^2$ is integrable on σ_ρ. For each of these ρ, there holds the equality

$$\int_{\Omega_\rho} \bar{v}\, \tilde{g}u\, dx + \int_{\sigma_\rho} \bar{v} \sum_{k,\ell=1}^{m} a_{k\ell}\alpha_k \frac{\partial u}{\partial x_\ell}\, d\sigma = E_\rho(u,v), \tag{11}$$

where $\alpha_k = \frac{x_k}{|x|}$. Equality (11) is first proved for functions v which vanish near Γ_ρ, and then in the general case. It follows from inequality (5) that:

$$\left| \int_{\sigma_\rho} \bar{v} \sum_{k,\ell=1}^{m} a_{k\ell}\alpha_k \frac{\partial u}{\partial x_\ell}\, d\sigma \right| \leq \frac{1}{2} \sqrt{M(\rho)} \int_{\sigma_\rho} \left[|v|^2 + \sum_{k,\ell=1}^{m} a_{k\ell} \frac{\partial u}{\partial x_k} \frac{\partial u}{\partial x_\ell} \right] d\sigma .$$

From this inequality and condition (6), it follows that there exists a sequence $\rho_n \longrightarrow \infty$ such that

$$\int_{\sigma_{\rho_n}} \bar{v} \sum_{k,\ell=1}^{m} a_{k\ell}\alpha_k \frac{\partial u}{\partial x_\ell}\, d\sigma \longrightarrow 0 \text{ as } n \longrightarrow \infty.$$

Passing to the limit in (11), we obtain the proof to our assertion.

We now put $F = P \cap \overset{o}{D}$ and define on F the operator G in the space $L_2(\Omega)$ with the aid of the equation $Gu = gu$. It follows from lemma 2 that G is a symmetric operator. We replace the boundary value problem (3), (2) by the equation

$$Gu = (\lambda+\varepsilon i)u + f. \tag{12}$$

It now follows from the symmetry of G that equation (12), where $\varepsilon \neq 0$, cannot have more than one solution.

LEMMA 3. *In each region* Ω_ρ *there exists a unique solution of equation* (3), *where* $\varepsilon \neq 0$, *belonging to the class* Φ_ρ.

Proof: The uniqueness is obvious. We will show that a solution exists. Let

$$|q(x)| \leq c_5$$

in Ω_ρ, where c_5 depends on ρ. Then there exists a sequence of continuously differentiable functions $f_n(x)$ and $q_n(x)$ in $\overline{\Omega}_\rho$ such that $f_n \longrightarrow f$ and $q_n \longrightarrow q$ in measure in Ω_ρ, and also

$$|q_n(x)| \leq c_5$$

(see [7]. We shall put

$$g_n u = g_o u + q_n u$$

and investigate the equation

$$g_n u = (\lambda+\varepsilon i)u + f_n. \tag{13}$$

With the aid of the method of finite differences it can be proven that for any n there exists a solution v_n of equation (13) which belongs to Φ_ρ (see [8]. From (13) it follows that:

$$\varepsilon \int_{\Omega_\rho} |v_n|^2 dx = - \Im \int_{\Omega_\rho} f\overline{v}_n dx,$$

which implies that the norm $||v_n||_{W_2^{(1)}(\Omega_\rho)}$ is uniformly bounded with respect to n. Then, according to a well-known theorem of V. I. Kondrashova [9], it is possible to select a subsequence v_{n_k} from v_n which converges in $L_2(\Omega_\rho)$ to some function v (henceforth the index k will be dropped).

Let $\phi_n = f_n - q_n v_n$, then in $L_2(\Omega_\rho)$ $\phi_n \longrightarrow \phi$, where $\phi = f - qv$. It follows from the equations

$$g_o v_n = (\lambda + \varepsilon i)v_n + \phi_n$$

that $||v_k - v_\ell||_{W_2^{(1)}(\Omega_\rho)} \longrightarrow 0$ as $k,\ell \longrightarrow \infty$. That is the same as $v_n \longrightarrow v$ in $W_2^{(1)}(\Omega_\rho)$ and $v \in \overset{o}{D}_\rho$. From (8) and (14) we have $||v_k - v_\ell||_{W_2^{(2)}(\omega')} \longrightarrow 0$ as $k,\ell \longrightarrow \infty$, where ω' is any compact subset of Ω_ρ. Then $v_n \longrightarrow v$ in $W_2^{(2)}(\omega')$, whence $v \in \Phi_\rho$. Passing to the limit in (14), we see that v satisfies equation (3) almost everywhere in Ω_ρ.

Q.E.D.

The function v constructed above depends on ρ. Therefore in what follows it will be denoted by $v^{(\rho)}$. We shall introduce the functions $u^{(\rho)}$ in Ω in the following manner:

$$u^{(\rho)} = v^{(\rho)} \text{ in } \Omega_\rho,$$

$$u^{(\rho)} = 0 \text{ in } \Omega - \Omega_\rho.$$

Then $u^{(\rho)} \in D$.

We pass to solving equation (12).

THEOREM 1. Let $\varepsilon \neq 0$. Then $u^{(\rho)} \longrightarrow u$ as $\rho \longrightarrow \infty$ in the sense of convergence in the space J and also in the sense of convergence in $W_2^{(2)}(\omega')$, where ω' is any finite interior subregion of the region Ω. The limit function also belongs to F and satisfies equation (12)[3].

Proof: As above, it is established that the norm of $u^{(\rho)}$ in $L_2(\Omega)$ in uniformly bounded with respect to ρ, and thereupon is also in the norm of $u^{(\rho)}$ in J. From that it follows that there exists a sequence $\rho_n \longrightarrow \infty$ such that $u^{(\rho_n)}$ converges in any selected region Ω_ρ in the sense of $L_2(\Omega_\rho)$ to some function u. Henceforth we will write u_n in place of $u^{(\rho_n)}$. Choose some $\rho > 0$. Let $u_{k\ell} = u_k - u_\ell$ and the numbers k and ℓ be so large that $\rho_k > \rho$ and $\rho_\ell > \rho$. Then, by virtue of lemma 1,

$$\int_{\Omega_\rho} \sum_{j=1}^{m} \left| \frac{\partial u_{k\ell}}{\partial x_j} \right|^2 dx \leq c_6 \int_{\Omega_{\rho+2}} [|u_{k\ell}|^2 + |g u_{k\ell}|^2] dx. \qquad (15)$$

The convergence of u_n in any region Ω_{ρ_o} in the sense of $W_2^{(1)}(\Omega_\rho)$ follows from (15). Consequently, $u \in D$. The boundedness of $||u_n||_J$ implies that $u \in J$.

Then in exactly the same way as in the proof of lemma 3, we have that u_n converges in any finite interior subregion ω' in the metric of $W_2^{(2)}(\omega')$. From that it follows that $u \in \overline{W}_2^{(2)}(\Omega)$ and (3) is satisfied almost everywhere in Ω. By the same token u belongs to F and satisfies equation (12).

It remains to prove that $u_n \longrightarrow u$ in J. The boundedness of $||u_n||_{L_2(\Omega)}$ and the equations

$$\varepsilon\int_\Omega |u_n|^2dx = \vartheta\int_\Omega u_n\bar{f}dx,$$

$$\varepsilon\int_\Omega |u|^2dx = \vartheta\int_\Omega u\bar{f}dx$$

imply that $u_n \longrightarrow u$ in $L_2(\Omega)$. Then it follows from the equations

$$E(u_n,u_n) = \int_{\Omega_{\rho_n}} \bar{u}_n\tilde{g}u_n dx,$$

$$E(u,u) = \int_\Omega \bar{u}\,\tilde{g}u dx$$

that $E(u_n,u_n) \longrightarrow E(u,u)$, which implies that $u_n \longrightarrow u$ in J. By virtue of the uniqueness of the solution of equation (12), we have $u^{(\rho)} \longrightarrow u$ as $\rho \longrightarrow \infty$.

Q.E.D.

The self-adjointness of the operator G follows from theorem 1. The self-adjointness of G in the case $g = -\Delta$, $\Omega = R_3$ was first proven in Carleman [10] and Friedrichs [11].

THEOREM 2. Let the functions $a_{k\ell}$, q satisfy the conditions enumerated above and, in addition, satisfy the inequalities

$$\left|\frac{\partial a_{k\ell}}{\partial x_\ell}\right| \leq c_7, \tag{16}$$

$$|a_{k\ell}| \leq c_8|x|. \tag{17}$$

in Ω. *Then each function* u, *satisfying the conditions* $u \varepsilon \Phi \cap L_2(\Omega)$, $gu \varepsilon L_2(\Omega)$, *belongs to* J.

Proof: We may consider the function u to be real. Put $v = u$ in equation (11). Then

$$E_\rho(u,u) = \int_{\Omega_\rho} u\, \tilde{g}u\, dx = B_\rho,$$

where

$$B_\rho = \int_{\sigma_\rho} \sum_{k,\ell=1}^{m} a_{k\ell}\alpha_k \frac{\partial u}{\partial x_\ell}\, d\sigma .$$

Obviously, it is sufficient to prove that

$$\varliminf_{\rho \to +\infty} B_\rho \leq 0.$$

Assuming the contrary, we have that for all $\rho \geq \rho_o$, $B_\rho \geq c > 0$ where c does not depend on ρ;

$$\int_{\rho_o}^{\rho} B_\rho d\rho = \frac{1}{2}\int_{\sigma_\rho} u^2 \sum_{k,\ell=1}^{m} a_{k\ell}\alpha_k\alpha_\ell d\sigma - \frac{1}{2}\int_{\sigma_{\rho_o}} u^2 \sum_{k,\ell=1}^{m} a_{k\ell}\alpha_k\alpha_\ell d\sigma$$

$$- \frac{1}{2}\int_{\Omega_\rho - \Omega_{\rho_o}} u^2 \sum_{k,\ell=1}^{m} \left(\alpha_k \frac{\partial a_{k\ell}}{\partial x_\ell} + a_{k\ell}\frac{\partial \alpha_k}{\partial x_\ell} \right) dx \leq c_9 \rho \int_{\sigma_\rho} u^2 d\sigma + c_{10}.$$

From this it follows that, for sufficiently large ρ,

$$\int_{\sigma_\rho} u^2 d\sigma \geq c_{11} > 0,$$

but that contradicts the fact that $u \in L_2(\Omega)$.

Q.E.D.

It follows from this theorem that if the coefficients $a_{k\ell}$ satisfy conditions (16) and (17), then the domain of definition F of the operator G is represented by the totality of all functions u satisfying the conditions $u \in \Phi \cap L_2(\Omega)$ and $gu \in L_2(\Omega)$. An analogous result in the case $g = -\Delta$, $\Omega = R_3$ was obtained by Friedrichs [11].

THE PRINCIPLE OF LIMITING ABSORPTION FOR A REGION WITH A FINITE BOUNDARY

In the present paragraph the classical existence and uniqueness theorems for the solution of the exterior Dirichlet problem for the Helmholtz equation are carried over to more general equations, considered in an infinite region with an arbitrary finite boundary. In this connection the above-mentioned result of A. Y. Povzner [3] is extended and sharpened. For the sake of simplicity, we shall in Sections 2, 3 limit ourselves to the case m=3.

Let us assume that there exists a sphere S_{ρ_0} such that the boundary Γ lies within S_{ρ_0}, and outside of S_{ρ_0}

$$a_{k\ell} = \delta_{k\ell},$$

$$|q(x)| \leq c_{12}|x|^{-(2+\alpha)},$$

$$|f(x)| \leq c_{13}|x|^{-(3+\beta)},$$

where $\alpha > \frac{1}{6}$, $\beta > 0$.[9] It is well-known [12], that then all points $\lambda > 0$ are points of the continuous spectrum of the operator G, but for $\lambda < 0$ the spectrum is discrete. We shall denote by A the set of all positive values of λ which are points of the continuous spectrum of G.

LEMMA 4. *Let the function* $\phi \in L_2(\Omega)$ *and for* $\rho > \rho_0$ *there holds the inequality*

$$\int_{S_\rho} |\phi(x)|^2 dS \leq c_{14}\rho^{-(4+2h)},$$

where $h > 0$ *is a constant. Then for the function*

$$Q(x) = \int_\Omega \frac{e^{i\mu|x-y|}}{|x-y|}\phi(y)dy,$$

where $|x-y|$ *is the distance of the point* x *from the point* y, *there holds the inequality*

$$\int_{S_\rho} \left|\frac{\partial Q}{\partial r} - i\mu Q\right|^2 dS \leq c_{15}\rho^{-2h_1},$$

where

$$h_1 = \min\left\{1, \frac{2h}{2+h}\right\}, \quad r = |x|.$$

Proof: It follows from the inequality

$$\int_{\Omega-\Omega_\rho} |\phi(x)|dx \;{}^{2} \leq \int_{\Omega-\Omega_\rho} \frac{dx}{r^{2+h}} \int_{\Omega-\Omega_\rho} |\phi(x)|^2 r^{3+h} dx$$

that

$$\int_{\Omega-\Omega_\rho} |\phi(x)|dx \leq c_{16}\, \rho^{-h}. \tag{21}$$

Let the point x belong to the region $\Omega-\Omega_\rho$. Then

$$\left(\int_{\Omega-\Omega_\rho} \left|\frac{\phi(y)}{x-y}\right| dy\right)^2 \leq \int_{\Omega-\Omega_\rho} \frac{dy}{|x-y|^2|y|^{3+h}} \int_{\Omega-\Omega_\rho} |y|^{3+h}|\phi(y)|^2 dy.$$

But, as it is easily seen, there holds the inequality

$$\int_{\Omega-\Omega_\rho} \frac{dy}{|x-y|^2|y|^{3+h}} \leq \frac{c_{17}}{r^2\rho^h}, \tag{22}$$

whence

$$\int_{\Omega-\Omega_\rho} |\frac{|\phi(y)|}{|x-y|} dy \leq \frac{c_{18}}{r\rho^h}, \tag{23}$$

We now set $\rho = r^{\frac{2}{2+h}}$. Then

$$\frac{\partial Q}{\partial r} - i\mu Q = i\mu \int_{\Omega_\rho} \frac{e^{i\mu|x-y|}}{|x-y|}\left(\frac{\partial|x-y|}{\partial r} - 1\right)\phi(y)dy$$

$$- \int_{\Omega_\rho} \frac{e^{i\mu|x-y|}}{|x-y|^2}\frac{\partial|x-y|}{\partial r}\phi(y)dy + i\mu \int_{\Omega-\Omega_\rho} \frac{e^{i\mu|x-y|}}{|x-y|}\left(\frac{\partial|x-y|}{\partial r} - 1\right)\phi(y)dy$$

$$- \int_{\Omega-\Omega_\rho-K_x} \frac{e^{i\mu|x-y|}}{|x-y|^2}\frac{\partial|x-y|}{\partial r}\phi(y)dy - \int_{K_x} \frac{e^{i\mu|x-y|}}{|x-y|^2}\frac{\partial|x-y|}{\partial r}\phi(y)dy$$

$$\equiv \ell_1(x) - \ell_2(x) + \ell_3(x) - \ell_4(x) - \ell_5(x).$$

where K_x is sphere of unit radius with center at the point x. The estimate

$$\frac{\partial|x-y|}{\partial r} - 1 = O\left(\frac{|y|^2}{r^2}\right)$$

implies that

$$\ell_1(x) = O\left(r^{-\frac{3h+2}{h+2}}\right).$$

By virtue of (23), the same estimate holds for $\ell_3(x)$ and $\ell_4(x)$. Obviously, $\ell_2(x) = O\left(\frac{1}{r^2}\right)$. Then

$$\int_{S_r} |\ell_1+\ell_2+\ell_3+\ell_4|^2 DS = O\left(r^{-2h_1}\right).$$

Let us now consider $\ell_5(x)$. Let $\tilde{\Omega}_r = \Omega_{r+1} - \Omega_{r-1}$.

$$|\ell_5(x)|^2 \leq c_{19} \int_{\tilde{\Omega}r} \frac{|\phi(y)|^2}{|x-y|^{\frac{3}{2}}} dy,$$

$$\int_{S_r} |\ell_5(x)|^2 dS \leq c_{19} \int_{\tilde{\Omega}r} |\phi(y)^2| dy \int_{S_r} \frac{1}{|x-y|^{3/2}} dS_x,$$

$$\int_{S_r} \frac{1}{|x-y|^{\frac{3}{2}}} dS_x = \frac{4\pi r}{|y|} [(r+|y|)^{\frac{1}{2}} - |r-|y||^{\frac{1}{2}}] \leq c_{20} \sqrt{r} ,$$

whence

$$\int_{S_r} |\ell_5(x)|^2 dS = O\left(r^{-\left(\frac{7}{2} + 2h\right)}\right).$$

Q.E.D.

Lemma 4 refines a theorem of A. Y. Povsner ([3], pp. 141-143).

THEOREM 3. Let $\lambda \varepsilon A$. Then a solution of equation (10), satisfying the radiation condition

$$\lim_{\rho \to \infty} \int_{S_\rho} \left|\frac{\partial u}{\partial r} - i\sqrt{\lambda u}\right|^2 dS = 0, \tag{24}$$

is unique.

Proof: Let u be a solution of the equation

$$\overline{G}u = \lambda u, \tag{25}$$

satisfying condition (20) . We will first show that $u \varepsilon L_2(\Omega)$. Let $\rho > \rho_o$. Then, in view of the fact that $u \varepsilon \Phi$, we have:

$$\int_{S_\rho} \left(u\frac{\partial \overline{u}}{\partial r} - \overline{u}\;\frac{\partial u}{\partial r}\right) dS = 0.$$

This and the identity

$$i\sqrt{\lambda}\left(u\frac{\partial\bar{u}}{\partial r} - \bar{u}\quad\frac{\partial u}{\partial r}\right) = \left|\frac{\partial u}{\partial r}\right|^2 + \lambda|u|^2 - \left|\frac{\partial u}{\partial r} - i\sqrt{\lambda}\,u\right|^2$$

imply:

$$\int_{S_\rho}\left[\left|\frac{\partial u}{\partial r}\right|^2 + \lambda|u|^2\right]dS = T_\rho(u), \tag{26}$$

where

$$T_\rho(u) = \int_{S_\rho}\left|\frac{\partial u}{\partial r} - i\sqrt{\lambda}\,u\right|^2 dS.$$

It follows from (24) that:

$$\lim_{\rho\to\infty}\int_{S_\rho}\left[\left|\frac{\partial u}{\partial r}\right|^2 + \lambda|u|^2\right]dS = 0. \tag{27}$$

Let $x \in \Omega-\Omega_{\rho_o}$. With the aid of (27) it is not difficult to prove the equality

$$u(x) = Q_1(x) + Q_2(x), \tag{28}$$

where

$$Q_1(x) = \frac{1}{4\pi}\int_{S_{\rho_o}}\left[\frac{e^{i\sqrt{\lambda}|x-y|}}{|x-y|}\frac{\partial u(y)}{\partial\nu} - u(y)\frac{\partial}{\partial\nu}\left(\frac{e^{i\sqrt{\lambda}|x-y|}}{|x-y|}\right)\right]dS(y)$$

$$Q_2(x) = -\frac{1}{4\pi}\int_{\Omega-\Omega_{\rho_o}}\frac{e^{i\sqrt{\lambda}|x-y|}}{|x-y|}q(y)u(y)dy,$$

and ν is the normal to the sphere S_{ρ_o} directed toward the origin.

Obviously $Q_1(x) = O\left(\frac{1}{r}\right)$, $T_\rho(Q_1) = O\left(\frac{1}{\rho^2}\right)$. Moreover, by virtue of the fact that $q = O(r^{-(2+\alpha)})$, we have:

$$|Q_2(x)|^2 \leq c_{21} \int_{\Omega-\Omega_{\rho_o}} \frac{1}{|x-y|^2|y|^{3+\alpha}}\,dy \int_{\Omega-\Omega_{\rho_o}} \frac{|u(y)|^2}{|y|^{1+\alpha}}\,dy \leq \frac{c_{22}}{r^2}\ ,$$

whence $Q_2(x) = O\left(\frac{1}{r}\right)$ and, consequently, $u(x) = O\left(\frac{1}{r}\right)$. We set

$$\tau_\rho(u) = \int_{S_\rho} |u|^2 dS.$$

We shall introduce the sequence γ_k in the following manner:

$$\gamma_o = 0,\ \gamma_{k+1} = \frac{2(\gamma_k+\alpha)}{2+\gamma_k+\alpha}\ ,$$

and put $\gamma = \frac{1}{2}[\sqrt{\alpha^2+8\alpha} - \alpha]$. It is not hard to prove that $\gamma_k < \gamma$, $\gamma_k < \gamma_{k+1}$. It follows from these inequalities that $\gamma_k \longrightarrow \gamma$, as $k \longrightarrow \infty$. Since $\alpha > \frac{1}{6}$, then $\gamma > \frac{1}{2}$. Without loss of generality we may assume that $\alpha < 1$. Then $\gamma < 1$, whence $\gamma_k < 1$.

We shall prove now that, for any k, there holds the relation

$$\tau_\rho(u) = O(\rho^{-2\gamma_k}). \qquad (29)$$

For $k = 0$ (29) is proved. Let us assume that for some k (29) is correct. Then, setting $\Phi = qu$, we obtain: $\tau_\rho(\Phi) = O(\rho^{-(4+2\alpha+2\gamma_k)})$. By virtue of lemma 4, $T_\rho(Q_2) = O(\rho^{-2\gamma_{k+1}})$, whence it follows: $T_\rho(u) = O(\rho^{-2\gamma_{k+1}})$. By virtue of (26), $\tau_\rho(u) = O(\rho^{-2\gamma_{k+1}})$. Hence the proof of (29) is completed.

Taking k sufficiently large, we have from (29), that $\tau_\rho(u) = O(\rho^{-\delta})$, where $\delta > 1$. Thus $u \varepsilon L_2(\Omega)$. By virtue of theorem 2, $u \varepsilon F$, whence, since $\lambda \varepsilon A$, it follows that $u = 0$.

Q.E.D.

We now pass to the justification of the principle of limiting absorption. We will prove that there exists a solution of equation (10) satisfying condition (24).

THEOREM 4. Let $\lambda \varepsilon A$. Then the solution u_ε of equation (12) tends to a solution u of equation (10) in the sense (Φ) as $\varepsilon \longrightarrow +0$, and u satisfies condition (24). Moreover, $u_\varepsilon \longrightarrow u$ uniformly in any finite interior subregion.[4]

Proof: We will establish first that the integral

$$H(u_\varepsilon) = \int_\Omega \frac{|u_\varepsilon|^2}{1+|x|^{1+\alpha}}\,dx$$

is uniformly bounded with respect to ε, provided $1 \geq \varepsilon > 0$. Let us assume the contrary. Then for some sequence $\varepsilon_n \longrightarrow +0$, $H(u_{\varepsilon_n}) \longrightarrow \infty$. In what follows we shall drop the index n. We put

$$v_\varepsilon = [H(u_\varepsilon)]^{-1/2} u_\varepsilon .$$

Then

$$H(v_\varepsilon) = 1, \tag{30}$$

$$gv_\varepsilon = (\lambda+\varepsilon i)v_\varepsilon + \psi_\varepsilon, \tag{31}$$

where $\psi_\varepsilon = [H(u_\varepsilon)]^{-1/2} f$. It follows from (30) that

$$\int_{\Omega_\rho} |v_\varepsilon|^2 dx \leq 1 + \rho^{1+\alpha}. \tag{32}$$

From lemma 1 and inequality (32) we have

$$\int_{\Omega_\rho} \sum_{k=1}^{3} \left|\frac{\partial v_\varepsilon}{\partial x_k}\right|^2 dx \leq c_{23}(\rho). \tag{33}$$

It follows from (32) and (33) that it is possible to select a subsequence from the sequence v_ε which converges in any region Ω_ρ in the sense of $L_2(\Omega_\rho)$ to some function v. This subsequence we will also denote by v_ε. By virtue of inequality (8) and lemma 1, the subsequence v_ε converges in the sense (Φ); consequently, $v \in \Phi$. From (31) we have

$$\overline{G}v = \lambda v. \tag{34}$$

Set

$$\sqrt{\lambda+\varepsilon i} = \eta + \sigma i,$$

where $\sigma > 0$ (then $\eta > 0$ also). Obviously $\eta \longrightarrow \sqrt{\lambda}$ and $\sigma \longrightarrow 0$ as $\varepsilon \longrightarrow +0$. Now let $x \in \Omega - \Omega_\rho$. Then it follows from (31), the condition $a_{k\ell} = \delta_{k\ell}$ in $\Omega - \Omega_{\rho_o}$, and the inclusion $v_\varepsilon \in F \subset J \subset W_2^{(1)}(\Omega)$, that

$$v_\varepsilon(x) = M_\varepsilon(x) + N_\varepsilon(x) + Q_\varepsilon(x), \tag{35}$$

where

$$M_\varepsilon(x) = \frac{1}{4\pi}\int_{S_{\rho_o}} \left[\frac{e^{(-\sigma+\eta i)|x-y|}}{|x-y|}\frac{\partial v\ (y)}{\partial \nu} - v_\varepsilon(y)\frac{\partial}{\partial \nu}\left(\frac{e^{(-\sigma+\eta i)|x-y|}}{|x-y|}\right)\right]dS_{(y)},$$

$$N_\varepsilon(x) = \frac{1}{4\pi}\int_{\Omega-\Omega_{\rho_o}} \frac{e^{(-\sigma+\eta i)|x-y|}}{|x-y|}\psi_\varepsilon(y)dy,$$

$$Q_\varepsilon(x) = -\frac{1}{4\pi}\int_{\Omega-\Omega_{\rho_o}} \frac{e^{(-\sigma+\eta i)|x-y|}}{|x-y|}q(y)v_\varepsilon(y)dy.$$

In view of the boundedness of $||v_\varepsilon||_{W_2^{(2)}(\omega)}$, where ω is any compact subset, by a theorem in [9] the norms $||v_\varepsilon||_{L_2(S_{\rho_o})}$, $||\frac{\partial v_\varepsilon}{\partial \nu}||_{L_2(S_{\rho_o})}$ are also bounded. Let $\rho_1 > \rho_o$. Then for a point x belonging to $\Omega-\Omega_{\rho_1}$, there holds the inequality

$$|M_\varepsilon(x)| \le \frac{c_{24}}{|x|},$$

where c_{24} does not depend on x and ε. Moreover, it follows from (19), (20) and (30) that

$$|Q_\varepsilon(x)|^2 \leq c_{25} \int_{\Omega-\Omega_{\rho_o}} \frac{dy}{|x-y|^2|y|^{3+\alpha}} \int_{\Omega-\Omega_{\rho_o}} \frac{|v_\varepsilon(y)|^2}{|y|^{1+\alpha}} dy \leq \frac{c_{26}}{|x|^2} .$$

$$|N_\varepsilon(x)| \leq c_{27} \int_{\Omega-\Omega_{\rho_o}} \frac{dy}{|x-y|\cdot|y|^{3+\beta}} = \frac{c_{28}}{|x|} .$$

From (35) we conclude

$$|v_\varepsilon(x)| \leq \frac{c_{29}}{|x|} . \tag{36}$$

It follows from embedding theorems for three-dimensional space that $v_\varepsilon \to v$ uniformly in any finite interior subregion. Consequently, the function v(x) also satisfies inequality (36). We see from (36) that uniform convergence holds in any finite interior subset. Moreover, it follows from (36) also that $H(v_\varepsilon - v) \to 0$, hence

$$H(v) = 1. \tag{37}$$

Let us put

$$M(x) = \frac{1}{4\pi} \int_{S_{\rho_o}} \left[\frac{e^{i\sqrt{\lambda}|x-y|}}{|x-y|} \frac{\partial v(y)}{\partial \nu} - v(y)\frac{\partial}{\partial \nu}\left(\frac{e^{i\sqrt{\lambda}|x-y|}}{|x-y|}\right)\right] dS_{(y)} .$$

$$Q(x) = -\frac{1}{4\pi} \int_{\Omega-\Omega_{\rho_o}} \frac{e^{i\sqrt{\lambda}|x-y|}}{|x-y|} q(y)v(y)dy.$$

Inequality

$$|Q_\varepsilon(x) - Q(x)| \leq c_{30} \int_{\Omega-\Omega_{\rho_0}} \frac{|v_\varepsilon - v|}{|x-y| \cdot |y|^{2+\alpha}}\, dy$$

$$+ c_{31}| - \sigma + \eta i - \sqrt{\lambda} i| \int_{\Omega-\Omega_{\rho_0}} \left| \frac{e^{(-\sigma+\eta i - \sqrt{\lambda} i)|x-y|} - 1}{(-\sigma+\eta i - \sqrt{\lambda} i)|x-y|} \right| \frac{1}{|y|^{3+\alpha}}\, dy$$

implies that $Q_\varepsilon(x) \longrightarrow Q(x)$ for any point $x \in \Omega - \Omega_{\rho_0}$. Analogously we obtain $M_\varepsilon(x) \longrightarrow M(x)$ and $N_\varepsilon(x) \longrightarrow 0$. It follows from (35) that

$$u(x) = M(x) + Q(x).$$

From this and lemma 4 we conclude that the function v satisfies condition (24). Theorem 3 and (34) imply that $v = 0$ in Ω, but this is a contradiction to equation (37). Thus the inequality

$$H(u_\varepsilon) \leq c_{32},$$

is proven. Using this inequality, we now apply to u_ε the same argument that was applied above to v_ε. Hence it is possible to select subsequence u_{ε_n} from u_ε, where $\varepsilon_n \longrightarrow +0$, converging in the sense $(\underline{\Phi})$, and even uniformly on any infinite interior subset, to some function u satisfying equation (10) and condition (24). It follows from theorem 3 that not only $u_{\varepsilon_n} \longrightarrow u$, but also $u_\varepsilon \longrightarrow u$ for $\varepsilon \longrightarrow +0$.

Q.E.D.

Remarks.

1) It follows from our argument that $u_\varepsilon \to u$ in the sense the metric defined by the integral $H(u)$.

2) Let us denote the function constructed above by $u_+(x)$. We could have considered $\varepsilon \longrightarrow -0$; then we would have obtained in the limit another solution $u_-(x)$ of equation (10) satisfying the radiation condition

$$\lim_{\rho \longrightarrow \infty} \int_{S_\rho} \left| \frac{\partial u}{\partial r} + i \sqrt{\lambda}\, u \right|^2 dS = 0.$$

3) We shall denote the solution of the equation

$$Gu = (\mu + \varepsilon i)u + f$$

by $u_\varepsilon^{(\mu)}(x)$. Then, with the aid of the same argument, it is not difficult to prove that if $\lambda \varepsilon A$, then $u_\varepsilon^{(\mu)} \longrightarrow u_+$ as $\varepsilon \longrightarrow +0$, $\mu \longrightarrow \lambda$. Analogously we have $u_\varepsilon^{(\mu)} \longrightarrow u_-$ as $\varepsilon \longrightarrow -0$, $\mu \longrightarrow \lambda$.

4) Let all points of the interval $a \leq \lambda \leq b$ belong to A. Then for each $\lambda \varepsilon [a,b]$ there exist the functions u_+ and u_-. These functions depend on x, λ, where $x \varepsilon \Omega$, $\lambda \varepsilon [a,b]$. Therefore in what follows we shall write $u_+ = u_+(x,\lambda)$, $u_- = u_-(x,\lambda)$. Reasoning just as in the proof of theorem 4, we obtain the inequality

$$\int_{\Omega} \frac{|u_{\pm}(x,\lambda)|^2}{1+|x|^{1+\alpha}}\, dx \leq c_{33}. \tag{38}$$

It is clear that $u_{\pm}(x,\lambda) \longrightarrow u_{\pm}(x,\lambda_o)$ as $\lambda \longrightarrow \lambda_o$ in the sense (Φ), and even uniformly in any finite interior subregion. From this it follows that $u_{\pm}(x,\lambda)$ is continuous in $\Omega x[a,b]$.

5) Let $u_{\varepsilon}(x,\lambda)$ be the solution of equation (12), considered as a function of $x \in \Omega$ and $\lambda \in [a,b]$. It follows from remark 3 that, as $\varepsilon \longrightarrow \pm 0$, $u_{\varepsilon}(x,\lambda) \longrightarrow u_{\pm}(x,\lambda)$ uniformly with respect $\lambda \in [a,b]$, $x \in \omega$, where ω is any interior subregion of Ω.

6) Let E_{λ} be the resolution of the identity generated by the operator G. Let the interval $[a,b] \subset A$. It then follows from the integral representation of the resolvent, the Stieltjes inversion formula, and remark 5, as in A. Y. Povzner [3] (pp. 152-153), that

$$\frac{\partial}{\partial\lambda}(E_{\lambda}f) = \frac{1}{2\pi i}\,\theta(x,\lambda), \tag{39}$$

where $\theta(x,\lambda) = u_{+}(x,\lambda) - u_{-}(x,\lambda)$, $x \in \Omega$, $\lambda \in [a,b]$.

7) Let $q \geq 0$ in Ω and the interval $(0,b]$ be contained in A. Then for all $\lambda \in (0,b]$

$$H(u_{\pm}(x,\lambda)) \leq c_{34}. \tag{40}$$

This inequality is proved by the same method as was used in theorem 4. In this context, in place of theorem 3 the following easily seen assertion is employed: if $q \geq 0$ in Ω, then every solution of the equation $\overline{G}v = 0$ satisfying the condition $v = O(|x|^{-1})$, $\frac{\partial v}{\partial x_k} = O(|x|^{-2})$ at infinity is identically zero. Inequality (40) implies that, for any finite interior subregion ω, there exists a constant c_{35} such that

$$|u_{\pm}(x,\lambda)| \leq c_{35}, \tag{41}$$

for all $x \in \omega$ and $\lambda \in (0,b]$. Moreover, it can be proven that $u_+(x,\lambda) \longrightarrow v(x)$, $u_-(x,\lambda) \longrightarrow v(x)$ as $\lambda \longrightarrow +0$, where $v(x)$ is the solution of the equation

$$\overline{G}v = f,$$

satisfying the conditions $v = O(|x|^{-1})$, $\frac{\partial v}{\partial x_k} = O(|x|^{-2})$.

8) Let the coefficients $a_{k\ell}$ be three-times continuously differentiable and $q(x)$ be continuously differentiable in Ω. Let, in addition, $q(x)$ depend only on $|x|$ in $\Omega - \Omega_{\rho_o}$[10]. Then $A = (0,\infty)$. In fact, let $\lambda > 0$ and u be a solution of the equation

$$Gu = \lambda u.$$

Then the function u is twice continuously differentiable in Ω (see S. G. Mikhlin [13]). It follows from an argument of A. Y. Povzner ([3], pp. 153-154) that $u = 0$ in $\Omega - \Omega_{\rho_o}$. Then by virtue of the uniqueness of the solution of the Cauchy problem for an elliptic equation, proved by E. M. Landü [14], we have $u = 0$ in Ω. By the same token $\lambda \varepsilon A$.

THEOREM 5. Let the interval $a \leq \lambda \leq b$ be contained in A. Then for any finite interior subregion ω there exists a constant c_{36} such that

$$|u_{\pm}(x,\lambda) - u_{\pm}(x,\mu)| \leq c_{36} \; |\lambda-\mu|.$$

for any λ, μ in $[a,b]$, and for all $x \varepsilon \omega$.

Proof: We will carry out the proof for the function $u_{+}(x,\lambda)$, which we shall simply denote by $u(x,\lambda)$.

We shall introduce the notation

$$\phi_{\lambda\mu}(x) = \frac{u(x,\mu)-u(x,\lambda)}{\mu-\lambda}$$

$$H_{o}(\phi_{\lambda\mu}) = \int_{\Omega} \frac{|\phi_{\lambda\mu}|^{2}}{1+|x|^{3+\alpha}} \, dx,$$

and prove that

$$H_{o}(\phi_{\lambda\mu}) \leq c_{37}, \tag{42}$$

for all λ,μ in $[a,b]$. Let us assume the contrary. Then there exists sequences $\lambda_{n} \longrightarrow p$, $\mu_{n} \longrightarrow p$ such that $H_{o}(\phi_{\lambda_{n}\mu_{n}}) \longrightarrow \infty$ (henceforth the index n is dropped). Set

$$\psi_{\lambda\mu} = [H_o(\phi_{\lambda\mu})]^{-1/2}\phi_{\lambda\mu}.$$

Then

$$\overline{G}\psi_{\lambda\mu} = \mu\psi_{\lambda\mu} + [H_o(\phi_{\lambda\mu})]^{-1/2}\, u(x,\lambda), \tag{43}$$

$$H_o(\psi_{\lambda\mu}) = 1. \tag{44}$$

It follows from (43), (44) that it is possible to select a subsequence (we will denote it also by $\psi_{\lambda\mu}$) from the sequence $\psi_{\lambda\mu}$ which converges in the sense (Φ) to some function ψ. From (43) we have

$$\overline{G}\psi = p\psi. \tag{45}$$

Let $x \in \Omega\text{-}\Omega_{\rho_o}$. Then

$$\psi_{\lambda\mu}(x) = \frac{1}{4\pi}\int_{S_{\rho_o}} \left[\frac{e^{i\sqrt{\mu}|x-y|} - e^{i\sqrt{\lambda}|x-y|}}{(\mu-\lambda)|x-y|\sqrt{H_o(\phi_{\lambda\mu})}}\,\frac{\partial u(y,\mu)}{\partial\nu} + \frac{e^{i\sqrt{\lambda}|x-y|}}{|x-y|}\,\frac{\partial\psi_{\lambda\mu}(y)}{\partial\nu}\right]dS_{(y)}$$

$$-\frac{1}{4\pi}\int_{S_{\rho_o}} \left[\frac{\partial}{\partial\nu}\left(\frac{e^{i\sqrt{\mu}|x-y|} - e^{i\sqrt{\lambda}|x-y|}}{(\mu-\lambda)|x-y|\sqrt{H_o(\phi_{\lambda\mu})}}\right)u(y,\mu) + \frac{\partial}{\partial\nu}\left(\frac{e^{i\sqrt{\lambda}|x-y|}}{|x-y|}\right)\psi_{\lambda\mu}(y)\right]dS_{(y)}$$

$$-\frac{1}{4\pi\sqrt{H_o(\phi_{\lambda\mu})}}\int_{\Omega-\Omega_{\rho_o}} \frac{e^{i\sqrt{\mu}|x-y|} - e^{i\sqrt{\lambda}|x-y|}}{(\mu-\lambda)|x-y|}\,u(y,\mu)q(y)\,dy$$

$$-\frac{1}{4\pi}\int_{\Omega-\Omega_{\rho_o}} \frac{e^{i\sqrt{\lambda}|x-y|}}{|x-y|}\,\psi_{\lambda\mu}(y)q(y)\,dy \equiv P_1(x) + P_2(x) + P_3(x) + P_4(x). \tag{46}$$

It follows from the convergence of $u(x,\lambda) \longrightarrow u(x,p)$ and $\psi_{\lambda\mu} \longrightarrow \psi$ in the sense (Φ), by virtue of an embedding theorem, that the norms $\left\|\frac{\partial u(y,\mu)}{\partial \nu}\right\|_{L_2(S_{\rho_o})}$ and $\left\|\frac{\partial \psi_{\lambda\mu}}{\partial \nu}\right\|_{L_2(S_{\rho_o})}$ are bounded.

From this and the inequality

$$\left|\frac{e^{i\sqrt{\mu}|x-y|} - e^{i\sqrt{\lambda}|x-y|}}{(\mu-\lambda)|x-y|}\right| \leq c_{38},$$

which is valid as long as $\lambda \geq a > 0$, $\mu \geq a > 0$, it follows that

$$|P_1(x)| \leq c_{39},$$

in the region $\Omega - \Omega_{\rho_1}$, where $\rho_1 > \rho$. Analogously, in the same region

$$|P_2(x)| \leq c_{40}.$$

Inequality (38) implies

$$|P_3(x)| \leq c_{41}\,[H_o(\phi_{\lambda\mu})]^{-1/2}.$$

Finally,

$$|P_4(x)|^2 \leq H_o(\psi_{\lambda\mu}) \int_{\Omega-\Omega_{\rho_o}} \frac{dy}{|x-y|^2|y|^{1+\alpha}} \leq c_{42}.$$

By the same token

$$|\psi_{\lambda\mu}(x)| \leq c_{43}, \qquad (47)$$

$$H_o(\psi) = 1. \tag{48}$$

Passing to the limit in (46), we obtain

$$\psi(x) = \frac{1}{4\pi} \int_{S_{\rho_o}} \left[\frac{e^{i\sqrt{p}|x-y|}}{|x-y|} \frac{\partial \psi(y)}{\partial \nu} - \psi(y) \frac{\partial}{\partial \nu} \left(\frac{e^{i\sqrt{p}|x-y|}}{|x-y|} \right) \right] dS_{(y)}$$

$$- \frac{1}{4\pi} \int_{\Omega - \Omega_{\rho_o}} \frac{e^{i\sqrt{p}|x-y|}}{|x-y|} q(y)\psi(y)dy. \tag{49}$$

From (47) and (49) we have

$$|\psi(x)| \leq c_{43}$$

and the estimate

$$\int_{\Omega - \Omega_{\rho_o}} \frac{dy}{|x-y| \cdot |y|^{2+\gamma}} = O(|x|^{-\gamma}),$$

where $0 < \gamma < 1$, which imply that $\psi(x) = O(|x|^{-1/6})$. This and (49) imply $\psi(x) = O(|x|^{-1/3})$. Iterating the argument, we obtain $\psi(x) = O(|x|^{-5/6})$. Then from (49), with the aid of lemma 4, we conclude

$$\lim_{\rho \to \infty} \int_{S_\rho} \left| \frac{\partial \psi}{\partial r} - i\sqrt{p}\psi \right|^2 dS = 0.$$

It follows from (45) and theorem 3 that $\psi = 0$ in Ω. But this contradicts equation (48). Thus inequality (42) is proven.

With the aid of an embedding theorem, it follows from inequalities (42), (8) and the equation

$$g\psi_{\lambda\mu} = \psi_{\lambda\mu} + u(x,\lambda)$$

that

$$|\psi_{\lambda\mu}(x)| \leq c_{39}(\omega)$$

in any finite interior subregion ω.

Q.E.D.

THE PRINCIPLE OF LIMITING AMPLITUDES

We shall now consider another principle with which it is possible to select a definite solution to the problem under consideration - the principle of limiting amplitudes. According to this principle, the solution of the problem (1), (2) is defined as the limit

$$\lim_{t \to +\infty} w(x,t)e^{i\sqrt{\lambda}t},$$

here $w(x,t)$ is the solution of the mixed boundary value problem for the equation

$$\frac{\partial^2 w}{\partial t^2} + gw = e^{-i\sqrt{\lambda}t}f(x) \qquad (50)$$

having zero initial and boundary values. This principle was formulated by A. N. Tikhonov and A. A. Samarskii [15] and justified in the case of $g = -\Delta$, $q = 0$, $\Omega = R_3$. Several other cases were considered by A. G. Suesknikov [1]. O. A. Ladyzhenskaya [16] carried out the proof in the case $g = -\Delta + q$, $\Omega = R_3$. The operator G was assumed not to have discrete eigenvalues. O. A. Ladyzhenskaya showed that even if the operator G has only one discrete eigenvalue, the limit written above does not in general exist. This circumstance diminishes the value of the principle of limiting amplitudes in comparison with the principle of limiting absorption since, for the validity of the latter, it is essential only that the number λ does not belong to the discrete spectrum of the operator G. However there is considerable interest in the question of the behavior of the solution of problem (50) as $t \longrightarrow \infty$; that is, the question of whether it is possible to obtain steady-state oscillations from transient oscillations by passing to the limit.

In the present paragraph we shall carry out the proofs corresponding to these assertions, retaining the assumptions of the previous paragraph. We shall also introduce several additional assumptions guaranteeing the absence of discrete eigenvalues of the operator G.

Instead of the above classically formulated problem (50), we shall consider the problem of determining a solution of the equation

$$\frac{d^2w}{dt^2} + Gw = fe^{-i\sqrt{\lambda}t}, \tag{51}$$

satisfying the initial conditions

$$w\Big|_{t=0} = \zeta_1, \ \frac{dw}{dt}\Big|_{t=0} = \zeta_2. \tag{52}$$

Here $w = w(t)$ is a function of t, $0 < t < \infty$, which is an element of the space $L_2(\Omega)$ and has the continuous derivatives $\frac{dw}{dt}$ and $\frac{d^2w}{dt^2}$.

Let the operator G be strictly positive, $\zeta_1 \varepsilon F$, and ζ_2 belong to the domain of definition of the operator $\sqrt{G}$, which we will henceforth denote by F_1. Then (see [17]) there exists a unique solution of the problem (51), (52). Moreover this solution can be obtained with the aid of the method of Fourier.

In the present paragraph we shall assume that all the conditions on Ω, $a_{k\ell}$, q, and f which were introduced in §2 are fulfilled. Let, in addition, the functions $a_{k\ell}$ be three-times continuously differentiable and q be once continuously differentiable in Ω. Further, let $q(x)$ depend only on $|x|$ outside of same sphere[10] and

$$q(x) \geq 0 \tag{53}$$

in Ω.

THEOREM 6. *Let* $\zeta_1 \varepsilon F$, $\zeta_2 \varepsilon F_1$, *and also*

$$|\zeta_k(x)| \leq c_{44}^{(k)} |x|^{-(3+\beta)}$$

in $\Omega - \Omega_o$, *where* $k = 1,2$, $\beta > 0$. *Then the solution of problem* (51), (52), *where* $\lambda > 0$, *is such that for each point*[5] $x \varepsilon \Omega$,

$$\lim_{t \to +\infty} w e^{i\sqrt{\lambda}t} = u_+(x,\lambda).$$

Moreover the convergence is uniform in any finite interior subregion ω.

The proof is based on the following lemma:

LEMMA 5. *Let the function* $\phi(x,\sigma)$ *be defined for* $a \leq \sigma \leq b$ *and* x *in some set* X. *Let*

$$|\phi(x,\sigma)| \leq c_{45}$$

for all $x \varepsilon X$, $\sigma \varepsilon [a,b]$, *and for all* $x \varepsilon X$ *and* $\sigma_1, \sigma_2 \varepsilon [a,b]$

$$|\phi(x,\sigma_2) - \phi(x,\sigma_1)| \leq c_{46} |\sigma_2 - \sigma_1|^{\gamma},$$

where $0 < \gamma \leq 1$. *Then for each number* λ, *such that* $a < \lambda < b$, *there holds*[6]

$$\lim_{t \to +\infty} e^{\lambda t i} \int_a^b \frac{e^{-\sigma t i}}{\sigma - \lambda} \phi(x,\sigma) d\sigma = -\pi i \phi(x,\lambda),$$

and the limit exists uniformly with respect to point $x \varepsilon X$.

The proof of this lemma follows easily from the equation

$$\lim_{t \to +\infty} e^{\lambda t i} \int_a^b \frac{e^{-\sigma t i}}{\sigma - \lambda} \, d\sigma = -\pi i.$$

Proof of Theorem 6. By virtue of inequality (53), the operator G is strictly positive. In addition, by remark (8) to theorem 4, we have: A = (0,∞). By the same token the spectral function E_λ of the operator G has no point of discontinuity. The solution of problem (51), (52), as it is not hard to verify, has the form:

$$w = \int_0^\infty \cos \sqrt{\sigma}\, t dE_\sigma \zeta_1 + \int_0^\infty \frac{\sin \sqrt{\sigma} t}{\sqrt{\sigma}} dE_\sigma \zeta_2$$
$$+ \int_0^\infty \frac{e^{-i\sqrt{\lambda} t} - \left(\cos \sqrt{\sigma} t - i\sqrt{\frac{\lambda}{\sigma}} \sin \sqrt{\sigma} t\right)}{\sigma - \lambda} dE_\sigma f. \qquad (54)$$

Let us consider the integral

$$U_1 = \int_0^\infty \cos \sqrt{\sigma} t dE_\sigma \zeta_1.$$

Obviously U_1 ε F for any t > 0; consequently (we are assuming m = 3), the function $U_1(x,t)$ is continuous with respect to x in Ω. We have:

$$U_1 = \int_0^N \cos\sqrt{\sigma}\, t dE_\sigma \zeta_1 + \int_N^\infty \cos\sqrt{\sigma}\, t dE_\sigma \zeta_1 \equiv U_1^{(1)} + U_1^{(2)},$$

$$||U_1^{(2)}||^2_{L_2(\Omega)} = \int_N^\infty \cos^2\sqrt{\sigma}\, t d(E_\sigma\zeta_1,\zeta_1) \leq \int_N^\infty d(E_\sigma\zeta_1,\zeta_1),$$

$$||GU_1^{(2)}||^2_{L_2(\Omega)} = \int_N^\infty \sigma^2\cos^2\sqrt{\sigma}\, t d(E_\sigma\zeta_1,\zeta_1) \leq \int_N^\infty \sigma^2 d(E_\sigma\zeta_1,\zeta_1).$$

We choose a number N so large that for all $t > 0$,

$$||U_1^{(2)}||_{L_2(\Omega)} < \varepsilon,$$

$$||GU_1^{(2)}||_{L_2(\Omega)} < \varepsilon.$$

Then, by virtue of (8),

$$||U_1^{(2)}||_{W_1^{(2)}(\omega)} \leq c_{47}(\omega)\varepsilon.$$

Hence by an embedding theorem [9], for $x \in \omega$, $t > 0$,

$$|U_1^{(2)}(x,t)| \leq c_{48}\varepsilon.$$

It follows from equation (39) that

$$U_1^{(1)}(x,t) = \frac{1}{2\pi i}\int_0^N \theta_1(x,\sigma)\cos\sqrt{\sigma}\, t d\sigma\,, \tag{55}$$

where $\theta_1(x,\sigma) = u_+^{(1)}(x,\sigma) - u_-^{(1)}(x,\sigma)$. In this $u_+^{(1)}(x,\sigma)$ is the solution of the equation

$$\overline{G}u = \sigma u + \zeta_1,$$

satisfying the radiation condition (24) and $u_-^{(1)}(x,\sigma)$ is the solution of the same equation satisfying the conjugate radiation condition.

From equation (55) we obtain, making use of remarks 4) and 7) to theorem 4, that $U_1^{(1)}(x,t) \longrightarrow 0$ as $t \longrightarrow \infty$ uniformly in the region ω. By the same token $U_1(x,t) \longrightarrow 0$ as $t \longrightarrow \infty$ uniformly in any interior subregion ω. By the same method it is not hard to prove an analogous result for the function

$$U_2 = \int_0^\infty \frac{\sin\sqrt{\sigma}\,t}{\sqrt{\sigma}}\, dE_\sigma \zeta_2 .$$

Let us now consider the integral

$$U_3 = \int_0^\infty \frac{e^{-i\sqrt{\lambda}t} - \left(\cos\sqrt{\sigma}\,t - i\sqrt{\frac{\lambda}{\sigma}}\sin\sqrt{\sigma}\,t\right)}{\sigma - \lambda}\, dE_\sigma f,$$

which is easily represented in the form

$$U_3 = \frac{1}{2\pi i}\left[\int_0^\infty \frac{\theta(x,\sigma)e^{-i\sqrt{\lambda}t}}{\sigma - \lambda}\,d\sigma - \int_0^\infty \frac{e^{-i\sqrt{\sigma}t}\theta(x,\sigma)}{\sigma - \lambda}\,d\sigma\right.$$

$$\left. - i\int_0^\infty \frac{\theta(x,\sigma)\sin\sqrt{\sigma}t}{(\sqrt{\sigma} + \sqrt{\lambda})\sqrt{\sigma}}\,d\sigma\right].$$

where $\theta(x,\sigma)$ is the function introduced in remark 6) to theorem 4. By virtue of theorem 5, this function satisfies a Hölder condition with respect to the variable σ. With the aid of lemma 5 and the same reasoning which was applied above to the integral U_1, we obtain that as $t \to \infty$,

$$e^{i\sqrt{\lambda}t}U_3(x,t) \to \frac{1}{2\pi i}\int_0^\infty \frac{\theta(x,\sigma)}{\sigma-\lambda}\,d\sigma + \frac{1}{2}\theta(x,\lambda)$$

uniformly in ω. It follows from (54) that

$$\lim_{t\to\infty} we^{i\sqrt{\lambda}t} = \frac{1}{2\pi i}\int_0^\infty \frac{\theta(x,\sigma)}{\sigma-\lambda}\,d\sigma + \frac{1}{2}\theta(x,\lambda), \tag{56}$$

and the convergence is uniform in any interior subregion ω.

Let $u_\varepsilon(x,\lambda)$ be the solution of equation (12), where $\varepsilon > 0$. It follows from the integral representation of the resolvent and equation (39) that

$$u_\varepsilon(x,\lambda) = \frac{1}{2\pi i}\int_0^\infty \frac{\theta(x,\sigma)}{\sigma-(\lambda+\varepsilon i)}\,d\sigma .$$

Passing to the limit as $\varepsilon \to +0$ and making use of the formula for the limiting value of an integral of the Cauchy type, we obtain

$$u_+(x,\lambda) = \frac{1}{2\pi i}\int_0^\infty \frac{\theta(x,\sigma)}{\sigma-\lambda}\,d\sigma + \frac{1}{2}\theta(x,\lambda).$$

The assertion of theorem 6 follows from this and (56).

Theorem 6 remains valid in the case $m = 2$. For $m = 1$ this theorem does not hold, as it is easy to see in the case case $g = - \frac{d^2}{dx^2}$, $\Omega = (-\infty,\infty)$.

We note that a result analogous to the assertion of theorem 6 can be obtained for an equation of the Schrödinger type

$$i \frac{\partial w}{\partial t} - gw = e^{-i\lambda t} f(x).$$

The Principle of Limiting Absorption for Cylindrical Tubes

We shall consider in the space R_m of the variables $x_1, x_2, \ldots, x_m$ a connected region Ω consisting of two parts Ω' and Ω''. The part Ω' is assumed to be finite (i.e. contained inside some sphere). The part Ω'' is represented by a semi-infinite cylinder $x_m > -\kappa$, $(x_1, x_2, \ldots, x_{m-1}, 0) \varepsilon \eta$, where $\kappa > 0$, η is a finite $(m-1)$-dimensional region, situated in the hyperplane $x_m = 0$ and having the boundary γ. We shall call such a region Ω a cylinderical tube. We will assume that the condition of §1 are fulfilled and that, in addition, $q = 0$, $f = 0$, $a_{k\ell} = \delta_{k\ell}$ in Ω''.

Let η_z be the intersection of the region Ω'' with the hyperplane $x_m = z$; γ_z the boundary of the region η_z; Ω_z the totality of all points of the region Ω for which $x_m < z$; Γ_z the totality of all points of Γ for which $x_m < z$; $\Omega_{z_1 z_2}$, where $-\kappa < z_1 < z_2$, the totality of all points of the region Ω'' contained between the hyperplanes $x_m = z_1$, $x_m = z_2$; and $\Gamma_{z_1 z_2}$ the totality of all points of Γ contained between the same hyperplanes. We shall introduce the class of functions $\overset{\circ}{D}{}^{(2)}_{z_1 z_2}$, where $-\kappa < z_1 < z_2$, as the completion in $W_2^{(1)}(\Omega_{z_1 z_2})$ of the set of functions defined in $\Omega_{z_1 z_2}$ and vanishing in boundary strip along $\Gamma_{z_1 z_2}$. Note that for all functions Φ, such that $\Phi \varepsilon \overset{\circ}{D}{}^{(2)}_{z_1 z_2} \cap \overline{W}_2^{(2)}(\Omega_{z_1 z_2})$,

$\Delta\phi \in L_2(\Omega_{z_1z_2})$, there holds the following inequality, analogous to inequality (9):

$$\int_{\Omega_{z_1'z_2'}} \sum_{k=1}^{m} \left|\frac{\partial\phi}{\partial x_k}\right|^2 dx \leq c_{49} \int_{\Omega_{z_1z_2}} [|\phi|^2 + |\Delta\phi|^2]dx, \tag{57}$$

where

$$z_1 < z_1' < z_2' < z_2, \; c_{49} = c_{49}(z_1,z_2,z_1',z_2').$$

Let now $\phi \in L_2(\Omega_{z_1z_2})$, $\frac{\partial\phi}{\partial x_m} \in L_2(\Omega_{z_1z_2})$. Then, as it is easily seen,

$$\int_{\eta_{z_i}} |\phi|^2 d\tilde{x} \leq \frac{2}{|z_2-z_1|} \int_{\Omega_{z_1z_2}} |\phi|^2 dx + 2|z_2-z_1| \int_{\Omega_{z_1z_2}} \left|\frac{\partial\phi}{\partial x_m}\right|^2 dx, \tag{58}$$

where

$$i = 1,2, \; -\kappa < z_1 < z_2, \; d\tilde{x} = dx_1dx_2,\dots,dx_{m-1}.$$

We shall denote by $\overset{\circ}{D}{}^{(2)}$ the totality of all functions ϕ which are defined in Ω'' and for which $\phi \in \overset{\circ}{D}{}^{(2)}_{z_1z_2}$ for any z_1,z_2, where $-\kappa < z_1 < z_2$.

Let μ_k and $\phi_k(\tilde{x})^7$ be the eigenvalues and eigenfunctions of the equation

$$\sum_{i=1}^{m-1} \frac{\partial^2\phi}{\partial x_i^2} + \mu\phi = 0 \tag{59}$$

in the region η with the initial condition $\phi|_{\eta} = 0$, regarded in the sense $\phi \in \overset{\circ}{D}(\eta)$. Here, as usual, $\mu_k < \mu_{k+1}$.

LEMMA 6. *Let* $v = v(\tilde{x}, x_m)$ *be the solution of the equation*

$$\Delta v + \tau v = 0$$

(τ *is a complex number*, $\tau \neq \mu_k$, $k=1,2,\ldots$) *in the region* Ω'', *satisfying the condition* $v \in \overset{\circ}{D}{}^{(2)}$, $\dfrac{\partial^\ell v}{\partial x_m^\ell} \in L_2(\Omega_{z_1 z_2})$, *where* $\ell=1,2,\ldots,\Lambda$, $\Lambda \geq 2$, z_1, z_2 *any numbers such that* $-\kappa < z_1 < z_2$. *Then the function* v, *considered on any cross-section* η_z, *is such that* $\dfrac{\partial^\ell v}{\partial x_m^\ell} \in L_2(\eta_z)$, *where* $\ell=1,2,\ldots,\Lambda-1$. *Moreover*, if

$$g_k(z) = \int_\eta v(\tilde{x}, z)\phi_k(\tilde{x})d\tilde{x},$$

where $z > -\kappa$, *it is possible to differentiate under the integral sign* $\Lambda-1$ *times, and*

$$g_k(z) = b_k' e^{\sqrt{\mu_k - \tau}\, z} + b_k'' e^{-\sqrt{\mu_k - \tau}\, z} \quad , \tag{60}$$

where b_k' *and* b_k'' *are constants*.

Proof. Inequality (58) implies that $\dfrac{\partial^\ell v}{\partial x_m^\ell} \in L_2(\eta_z)$ and each of the integrals $\displaystyle\int_{\eta_z} \left|\frac{\partial^\ell v}{\partial x_m^\ell}\right|^2 d\tilde{x}$ is uniformly bounded in any finite interval $z_1 \leq z \leq z_2$, where $\ell=1,2,\ldots,\Lambda-1$. This implies the uniform convergence of each of the integrals $\int_\eta \frac{\partial^\ell v}{\partial z^\ell} \phi_k \, d\tilde{x}$ $(\ell=1,2,\ldots,\Lambda-1)$ in any interval $z_1 \leq z \leq z_2$. By the same token

$$\frac{d^\ell g_k}{dz^\ell} = \int_\eta \frac{\partial^\ell v}{\partial z^\ell} \phi_k d\tilde{x},$$

where $\ell=1,2,\ldots,\Lambda-1$. Let $-\kappa < z_1 < z$. Then

$$\int_{\Omega_{z_1 z}} \phi_k \Delta v \, dx = \int_{\Omega_{z_1 z}} v \, \Delta\phi_k dx + g_k'(z) - g_k'(z_1),$$

whence

$$(\mu_k-\tau) \int_{\Omega_{z_1 z}} v\phi_k dx = g_k'(z) - g_k'(z_1).$$

Differentiating this identity with respect to z, we obtain

$$g_k''(z) = (\mu_k-\tau)g_k(z),$$

which implies (60).

Q.E.D.

LEMMA 7. *Let* $u(x) = u(\tilde{x},x_m)$ *be the solution of equation* (12), *where* $\varepsilon \neq 0$. *Then* $\frac{\partial^k u}{\partial x_m^k} \in \overset{\circ}{D}{}^{(2)}$, *where* $k=0,1,2,\ldots$ *Moreover*,

$$\int_{\eta} u(\tilde{x},z)\,\phi_k(\tilde{x})\,d\tilde{x} = b_k e^{-\sqrt{\mu_k-\lambda-\varepsilon i}\; z}, \qquad (61)$$

where $\Re\sqrt{\mu_k-\lambda-\varepsilon i} > 0$, and b_k is a constant.

For the proof of the lemma, let us first assume that the functions q and f are continuously differentiable in $\Omega + \Gamma$. By means of the method of finite differences, solutions of the equation $gv_z = (\lambda+\varepsilon i)v_z + f$ in the region Ω_z can be constructed which belong to the class $\Phi'_z = \overset{\circ}{D}(\Omega_z) \cap \overline{W}_2^{(2)}(\Omega_z)$. In view of the fact that the difference relation of any order taken with respect to x_m are equal to zero on that boundary strip of lattice points where $-\kappa < x_m < z$, it can be proven, using differences analogous to inequality (57), that for any z_1, z_2 such that $-\kappa < z_1 < z_2 < z$, $\frac{\partial^k v_z}{\partial x_m^k} \in \overset{\circ}{D}{}^{(2)}_{z_1 z_2}$. In this, the method of [8] is utilized. Reasoning in the same way as in the proof of lemma 3, it is possible to eliminate the restriction imposed on q and f.

We now put $u_z = v_z$ in Ω_z, $u_z = 0$ in Ω-Ω_z and let z tend to $+\infty$. Then according to theorem 1, $u_z \longrightarrow u$ in $W_2^{(1)}(\Omega)$ and, consequently, $\frac{\partial u_z}{\partial x_m} \longrightarrow \frac{\partial u}{\partial x_m}$ in $L_2(\Omega)$. Choose the numbers z_1 and z_2, $-\kappa < z_1 < z_2$, arbitrarily. It follows from inequality (57) $\frac{\partial u_z}{\partial x_m}$ converges in $W_2^{(1)}(\Omega_{z_1 z_2})$ as $z \longrightarrow \infty$, which implies that $\frac{\partial u}{\partial x_m} \in \overset{\circ}{D}{}^{(2)}$ and also that $\frac{\partial^2 u_z}{\partial x_m^2} \longrightarrow \frac{\partial^2 u}{\partial x_m^2}$ in $L_2(\Omega_{z_1 z_2})$. From the latter fact we obtain in an analogous manner that $\frac{\partial^2 u}{\partial x_m^2} \varepsilon \overset{o}{D}^{(2)}$ and $\frac{\partial^3 u_z}{\partial x_m^3} \longrightarrow \frac{\partial^3 u}{\partial x_m^3}$ in $L_2(\Omega_{z_1 z_2})$. By induction we have $\frac{\partial^k u}{\partial x_m^k} \varepsilon \overset{o}{D}^{(2)}$, where k=1,2,... . Lemma 6 and the fact that u belongs to $L_2(\Omega)$ implies equation (61).

Q.E.D.

It was shown in Jones [18] that the spectrum of the operator G is discrete to the left of the point μ_1 and continuous to the right of this point. Denote by A_1 the set of all points of continuity of the spectral function of the operator G which do not coincide with any of the points μ_k.

THEOREM 7. Let $\lambda \varepsilon A_1$, $\mu_p < \lambda < \mu_{p+1}$. Then there exists not more than one solution $u = u(\tilde{x}, x_m)$ of equation (10) for which $\frac{\partial u}{\partial x_m} \varepsilon L_2(\eta_z)$ and

$$\int_\eta u(\tilde{x},z)\phi_k(\tilde{x})d\tilde{x} = a_k e^{-\sqrt{\mu_k-\lambda}\, z}, \tag{62}$$

$$\int_\eta \frac{\partial u(\tilde{x},z)}{\partial z} \phi_k(\tilde{x})d\tilde{x} = - a_k \sqrt{\mu_k-\lambda}\, e^{-\sqrt{\mu_k-\lambda}\, z}, \tag{63}$$

where $z > -\kappa$, a_k is a constant, and $\sqrt{\mu_k-\lambda} = -i\sqrt{\lambda-\mu_k}$ for $1 \leq k \leq p$.

Proof. Consider a solution of the equation $\overline{G}u = \lambda u$, satisfying conditions (62) and (63). Expanding the functions u and $\frac{\partial u}{\partial x_m}$ in a Fourier series with respect to the ϕ_k in the cross-section η_z, we have:

$$u = \sum_{k=1}^{p} a_k e^{i\sqrt{\lambda-\mu_k}\, z} \phi_k + \sum_{k=p+1}^{\infty} a_k e^{-\sqrt{\mu_k-\lambda}\, z} \phi_k,$$

$$\frac{\partial u}{\partial x_m} = i \sum_{k=1}^{p} a_k \sqrt{\lambda-\mu_k}\, e^{i\sqrt{\lambda-\mu_k}\, z} \phi_k - \sum_{k=p+1}^{\infty} a_k \sqrt{\mu_k-\lambda}\, e^{-\sqrt{\mu_k-\lambda}\, z} \phi_k.$$

Substituting these expressions in the equation

$$\int_{\eta_z} \left(u \frac{\partial \overline{u}}{\partial x_m} - \overline{u} \frac{\partial u}{\partial x_m} \right) d\tilde{x} = 0,$$

we obtain

$$2i \sum_{k=1}^{p} |a_k|^2 \sqrt{\lambda-\mu_k} = 0,$$

whence $a_k = 0$ for $k=1,2,\ldots,p$. By the same token

$$u = \sum_{k=p+1}^{\infty} a_k e^{-\sqrt{\mu_k-\lambda}\, z} \phi_k,$$

in any cross-section η_z, where $z > -\kappa$, which implies that $u \varepsilon L_2(\Omega)$. By theorem 2, $u \varepsilon F$. Consequently, since $\lambda \varepsilon A_1$, then $u = 0$.

Q.E.D.

THEOREM 8. *Let* $\lambda \varepsilon A_1$. *Then the solution* u_ε *of equation* (12) *as* $\varepsilon \longrightarrow +0$ *tends in the sense* (Φ) *to a solution* $u = u(\tilde{x},x_m)$ *of equation* (10), *for which* $\frac{\partial u}{\partial x_m} \varepsilon L_2(\eta_z)$, *where* $z > \kappa$, *and which also satisfies equation* (62) *and* (63).

Proof. We shall prove first that, no matter how $z_o > 0$ is taken, the integral

$$h_{z_o}(u_\varepsilon) = \int_{\Omega_{z_o}} |u_\varepsilon|^2 dx$$

is uniformly bounded with respect to ε, where $1 \geq \varepsilon > 0$. We shall use the same argument as for the proof of theorem 4. Let us assume that for some sequence $\varepsilon_n \longrightarrow 0$, $h_{z_o}(u_{\varepsilon_n}) \longrightarrow \infty$, where z_o is fixed. Henceforth the index n will be dropped. We put

$$v_\varepsilon = |h_{z_o}(u_\varepsilon)|^{-1/2} u_z.$$

Then

$$h_{z_o}(v_\varepsilon) = 1,$$

$$gv_\varepsilon = (\lambda+\varepsilon i)v_\varepsilon + [h_{z_o}(u_\varepsilon)]^{-1/2} f.$$

It follows from this that it is possible to select a subsequence (we will denote it also by v_ε) from the sequence v_ε which converges in any region Ω_z, where $z < z_o$, in the sense $W_2^{(1)}(\Omega_z)$.

Then v_ε converges in each cross-section η_z, $z < z_o$, in the sense of $L_2(\eta_z)$. In particular, this is valid for $z = 0$, i.e. in the cross-section η_o. By virtue of lemma 7,

$$v_\varepsilon = \sum_{k=1}^{\infty} b_k(\varepsilon) e^{-\sqrt{\mu_k-\lambda-\varepsilon i}\, z} \phi_k, \tag{65}$$

in any cross-section η_z, $z > -\kappa$, where $\mathcal{R}\sqrt{\mu_k-\lambda-\varepsilon i} > 0$.

It follows from the convergence of v_ε in η_o in the sense of $L_2(\eta_o)$ that the elements $\left\{b_k(\varepsilon)\right\}_{k=1,2,\dots}$ of the space ℓ_2 tend, in the sense of convergence in that space, to some element $\left\{b_k\right\}_{k=1,2,\dots}$. As $\varepsilon \to +0$ and for $k=1,2,\dots,p$, where $\mu_p < \lambda < \mu_{p+1}$, we have $\sqrt{\mu_k-\lambda-\varepsilon i} \longrightarrow -\sqrt{\lambda-\mu_k}$. If $k > p$, then $\sqrt{\mu_k-\lambda-\varepsilon i} \longrightarrow \sqrt{\mu_k-\lambda}$. It follows, with the aid of (65), that v_ε converges to some function v in any region Ω_z in the sense $W_2^{(1)}(\Omega_z)$. Expanding the function v in any cross-section η_z, where $z > -\kappa$, with respect to the eigenfunctions ϕ_k, we have

$$v = \sum_{k=1}^{p} b_k e^{i\sqrt{\lambda-\mu_k}\, z} \phi_k + \sum_{k=p+1}^{\infty} b_k e^{-\sqrt{\mu_k-\lambda}\, z} \phi_k. \tag{66}$$

Further, $v_\varepsilon \longrightarrow v$ in the sense $(\underline{\Phi})$; consequently, the function v satisfies the equation

$$\overline{G}v = \lambda v. \tag{67}$$

By virtue of lemma 7, $\frac{\partial v_\varepsilon}{\partial x_m} \varepsilon \overset{o}{D}^{(2)}$. From the convergence $\frac{\partial v_\varepsilon}{\partial x_m} \rightarrow \frac{\partial v}{\partial x_m}$ in the sense $L_2(\Omega_{z,z_1})$, where z_1, z_2 are arbitrary numbers greater than $-\kappa$, and the inequality (57), we obtain that the sequence $\frac{\partial v_\varepsilon}{\partial x_m}$ converges in the sense of $W_2^{(1)}(\Omega_{z_1 z_2})$. Consequently, $\frac{\partial^2 v}{\partial x^2_m}$ belongs to $L_2(\Omega_{z_1 z_2})$.

By virtue of lemma 6, $\frac{\partial v}{\partial x_m} \varepsilon L_2(\eta_z)$ and the development of the function $\frac{\partial v}{\partial x_m}$ in the cross-section η_z in terms of the functions ϕ_k has the form:

$$\frac{\partial v}{\partial x_m} = i \sum_{k=1}^{p} b_k \sqrt{\lambda-\mu_k}\, e^{i\sqrt{\lambda-\mu_k}\, z} \phi_k - \sum_{k=p+1}^{\infty} b_k \sqrt{\mu_k-\lambda}\, e^{-\sqrt{\mu_k-\lambda}\, z} \phi_k. \quad (68)$$

It follows from (66), (67), (68), and theorem 7 that $v = 0$. But that contradicts equation (64). By the same token the inequality

$$\int_{\Omega_{z_o}} |u_\varepsilon|^2 dx \leq c_{50}(z_o),$$

is proven.

With the aid of this inequality, we prove, using the same argument, that it is possible to select a subsequence from u_ε which converges in the sense (Φ) to some solution u of equation (10), satisfying the conditions: $\frac{\partial u}{\partial x_m} \varepsilon L_2(\eta_z)$, (62), (63). Then, by theorem 7, $u_\varepsilon \rightarrow u$ in the sense (Φ).

Q.E.D.

THEOREM 9. *Let the interval* $a \le \lambda \le b$ *be contained in* A_1. *Let* $u(x,\lambda)$ *be the limit of* u_ε *as* $\varepsilon \longrightarrow +0$ *that was constructed above, considered as a function of* x *and* λ, *where* $x \in \Omega$, $\lambda \in [a,b]$. *Then, for any* z_0, *there exists a constant* $c_{51}(z_0)$ *such that*

$$\int_{\Omega_{z_0}} |u(x,\lambda_2) - u(x,\lambda_1)|^2 dx \le c_{51}|\lambda_2-\lambda_1|^{\delta}, \tag{69}$$

where $\delta = 2$, *holds for all* λ_1,λ_2 *in* $[a,b]$.

Proof: Let us introduce the following notation:

$$X_{\lambda_1\lambda_2}(x) = \frac{u(x,\lambda_1)-u(x,\lambda_2)}{\lambda_1-\lambda_2},$$

$$\psi_{\lambda_1\lambda_2} = [h_{z_0}(X\lambda_1\lambda_2)]^{-1/2}X_{\lambda_1\lambda_2}.$$

Assume that the asserted theorem is false. Then for some $z_0 > 0$ there exists subsequences $\lambda_1^{(n)}$ and $\lambda_2^{(n)}$, where $\lambda_1^{(n)},\lambda_2^{(n)} \in [a,b]$, such that $\lambda_1^{(n)} \longrightarrow \lambda, \lambda_2^{(n)} \longrightarrow \lambda$, $h_{z_0}(X_{\lambda_1^{(n)}\lambda_2^{(n)}}) \longrightarrow \infty$ (henceforth we drop the index n). For $\psi_{\lambda_1\lambda_2}$ there holds the equation

$$h_{z_0}(\psi_{\lambda_1\lambda_2}) = 1, \tag{70}$$

and the relation analogous to (43). This implies that it is possible to select a subsequence (we denote it also by $\psi_{\lambda_1\lambda_2}$)

from the sequence $\psi_{\lambda_1\lambda_2}$ which converges in the sense of $W_2^{(1)}(\Omega_{z_0})$. Then $\psi_{\lambda_1\lambda_2}$ converges in the cross-section η_0 in the sense $L_2(\eta_0)$. The development of function $\psi_{\lambda_1\lambda_2}$ in terms of the functions ϕ_k in the cross-section η_z, where z is any positive number, has the form:

$$\psi_{\lambda_1\lambda_2} = \sum_{k=1}^{\infty} P_k(\lambda_1,\lambda_2,z)\phi_k, \tag{71}$$

where[8]

$$P_k(\lambda_1,\lambda_2,z) = \frac{a_k(\lambda_1)e^{-\sqrt{\mu_k-\lambda_1}\,z} - a_k(\lambda_2)e^{-\sqrt{\mu_k-\lambda_2}\,z}}{(\lambda_1-\lambda_2)\sqrt{h_{z_0}(X_{\lambda_1\lambda_2})}},$$

$$a_k(\lambda_i) = e^{\sqrt{\mu_k-\lambda_i}\,z}\int_{\eta_z} u(x,\lambda_i)\phi_k(\tilde{x})d\tilde{x} \quad (i=1,2).$$

It follows from the convergence of the sequence $\psi_{\lambda_1\lambda_2}$ in η_0 in the sense $L_2(\eta_0)$ that the element $\left\{P_k(\lambda_1,\lambda_2,0)\right\}_{k=1,2,\ldots}$ of the space ℓ_2 tends in the sense of convergence in ℓ_2 to some element $\left\{P_k\right\}_{k=1,2,\ldots}$. We represent equation (71) in the form

$$\psi_{\lambda_1\lambda_2} = \psi^{(1)}_{\lambda_1\lambda_2} + \psi^{(2)}_{\lambda_1\lambda_2},$$

where

$$\psi^{(1)}_{\lambda_1\lambda_2} = \frac{1}{\sqrt{h_{z_o}(X_{\lambda_1\lambda_2})}} \sum_{k=1}^{\infty} a_k(\lambda_1) \frac{e^{-\sqrt{\mu_k-\lambda_1}\, z} - e^{-\sqrt{\mu_k-\lambda_2}\, z}}{\lambda_1-\lambda_2} \phi_k,$$

$$\psi^{(2)}_{\lambda_1\lambda_2} = \sum_{k=1}^{\infty} P_k(\lambda_1,\lambda_2,0) e^{-\sqrt{\mu_k-\lambda_2}\, z} \phi_k.$$

Note that the integral $\int_{\eta_o} |u(x,\lambda)|^2 d\tilde{x}$ is bounded uniformly for $\lambda \, \varepsilon[a,b]$. This implies that the sum $\sum_{k=1}^{\infty} |a_k(\lambda)|^2$ is bounded. From this it follows that $\psi^{(1)}_{\lambda_1\lambda_2} \longrightarrow 0$ in the sense of $L_2(\eta_z)$ in any cross-section η_z, and, moreover, that the convergence is uniform with respect to z in any finite interval. Analogously,

$$\psi^{(2)}_{\lambda_1\lambda_2} \longrightarrow \sum_{k=1}^{\infty} P_k e^{-\sqrt{\mu_k-\lambda}\, z} \phi_k,$$

in any η_z. By the same token the sequence $\psi_{\lambda_1\lambda_2}$ converges to some function ψ in any region Ω_z in the sense of $L_2(\Omega_z)$, and consequently, also in the sense $(\underline{\Phi})$. Moreover the development of ψ in the cross-section η_z has the form

$$\psi = \sum_{k=1}^{\infty} P_k e^{-\sqrt{\mu_k-\lambda}\, z} \phi_k. \tag{72}$$

We have:

$$\overline{G}\psi = \lambda\psi,$$

$$h_{z_o}(\psi) = 1. \tag{73}$$

It was established in the proof of theorem 8 that $\frac{\partial u_\varepsilon}{\partial x_m}$ converges as $\varepsilon \longrightarrow +0$ in the sense $W_2^{(1)}(\Omega_{z_1 z_2})$. Consequently $\frac{\partial u}{\partial x_m} \varepsilon \overset{o}{D}{}^{(2)}$, which implies that $\frac{\partial \psi_{\lambda_1 \lambda_2}}{\partial x_m} \varepsilon \overset{o}{D}{}^{(2)}$. From this, with the aid of inequality (57), we obtain that the sequence $\frac{\partial \psi_{\lambda_1 \lambda_2}}{\partial x_m}$ converges in the sense of $W_2^{(1)}(\Omega_{z_1 z_2})$, where z_1 and z_2 are any numbers such that $-\kappa < z_1 < z_2$. By the same token, $\frac{\partial^2 \psi}{\partial x_m^2} \varepsilon L_2(\Omega_{z_1 z_2})$. This implies that $\frac{\partial \psi}{\partial x_m} \varepsilon L_2(\eta_z)$, by lemma 6, and that the development of the function ψ in η_z, by virtue of (72), has the form:

$$\frac{\partial \psi}{\partial x_m} = - \sum_{k=1}^{\infty} P_k \sqrt{\mu_k - \lambda}\, e^{-\sqrt{\mu_k - \lambda}\, z} \phi_k .$$

Then, by theorem 7, $\psi = 0$ in Ω, which contradicts equation (73).

Q.E.D.

For the remainder, we need two lemmas. Let ω be an arbitrary finite region with boundary γ, and γ_o be some part of the boundary γ. Denote by $\overset{o}{D}_{\gamma_o}(\omega)$ the closure in the metric $W_2^{(1)}(\omega)$ of the set of functions defined in ω and vanishing in a boundary strip along γ_o. In addition, denote by $H_k(\omega)$ the set of functions defined in ω whose partial derivatives of order κ satisfy a Hölder condition. Let x be a point on γ. Denote by $K_\rho(x)$ the totality of all points of ω whose distance from x is less than ρ.

LEMMA 8. Let γ_o be a Lyapunov surface. Let w belong to the class $\overset{o}{D}_{\gamma_o}(\omega)$ and be the solution of the equation

$$\Delta w + \lambda w = 0. \tag{74}$$

Then for any interior point x_o of the set γ_o, there is a ρ such that $w \varepsilon H_1(K_\rho(x_o))$.

Proof. Let x_o be an interior point of γ_o. We shall construct a region ω_1, bounded by a closed Lyapunov surface γ_1, such that $\omega_1 \subset \omega$ and the surfaces γ_o and γ_1 have the common part γ_2, where x_o is an interior point of the set γ_2. We fix some $\rho > 0$ such that there is no point of the set $\gamma_1 - \gamma_2$ in the sphere of radius ρ. Then $K_\rho(x_o) \subset \omega_1$. Let $\xi_\rho(x)$ be some function, twice continuously differentiable in ω_1, such that

$$\xi_\rho(x) = 1 \text{ in } K_{\frac{\rho}{2}}(x_o),$$

$$\xi_\rho(x) = 0 \text{ in } \omega_1 - K_\rho(x_o).$$

Set $v_\rho = w\,\xi_\rho$. Then $v_\rho \;\varepsilon\; \overset{o}{D}(\omega_1)$ and the function v_ρ satisfies the equation

$$\Delta v_\rho = -\lambda v_\rho + \psi_\rho,$$

in ω_1, where $\psi_\rho = w\Delta\xi_\rho + 2\sum_{k=1}^{m} \frac{\partial\psi_\rho}{\partial x_k}\frac{\partial w}{\partial x_k}$. The function $\psi_\rho(x)$ in zero outside the region $M_\rho = K_\rho(x_o) - K_{\frac{\rho}{2}}(x_o)$.

Let $N(x,y)$ be the Green's function for the Laplace operator in the region ω_1. Then, as is easily seen,

$$v_\rho = -\lambda \int_{\omega_1} N(x,y)v_\rho(y)dy + \int_{\omega_1} N(x,y)\psi_\rho(y)dy.$$

Integrating k times and denoting the iterated kernel generated from $N(x,y)$ by $N_\ell(x,y)$, we obtain

$$v_\rho(x) = (-\lambda)^k \int_{\omega_1} N_k(x,y)v_\rho(y)dy + \sum_{\ell=1}^{k} (-\lambda)^{\ell-1} \int_{M_\rho} N_\ell(x,y)\psi_\rho(y)dy.$$

By setting $k = \left[\frac{m}{2}\right] + 1$ here and making use of the facts that $v_\rho \varepsilon L_2(\omega_1)$, $\psi_\rho \varepsilon L_2(\omega_1)$, we obtain that the function v_ρ, and consequently also the function w, is bounded in the region $K_{\frac{\rho}{4}}(x_o)$. From the equation

$$v_{\frac{\rho}{4}}(x) = -\lambda \int_{K_{\frac{\rho}{4}}(x_o)} N(x,y)v_{\frac{\rho}{4}}(y)dy + \int_{M_{\frac{\rho}{4}}} N(x,y)\psi_{\frac{\rho}{4}}(y)dy$$

and the estimates of the derivatives of the Green's function [19], it follows that the derivatives $\frac{\partial w}{\partial x_i}$ is bounded in the region $K_r(x_o)$, where $r = \frac{1}{16}\rho$.

It is well known (see [20], p. 289), that the function

$$F(x) = \int_{\omega_1} N(x,y)\nu(y)dy,$$

where $\nu(y)$ is bounded in ω, belongs to $H_1(\omega_1)$. It follows from the equation

$$v_r(x) = \int_{K_r(x_o)} N(x,y)\left[\psi_r(y) - \lambda v_r(y)\right]dy,$$

by virtue of the proven boundedness of $\psi_r - \lambda v_r$ in $K_r(x_o)$, that $v_r \varepsilon H_1(\omega_1)$. This implies $w \varepsilon H_1(K_{\frac{r}{2}}(x_o))$.

The proof of the lemma implies that the function w and its derivative $\frac{\partial w}{\partial x_i}$ are continuous up to γ_o, and also $w|_{\gamma_o} = 0$ in the classical sense.

LEMMA 9. *Let the surface* Γ_z *be a Lyapunov surface for any* z. *Let* w *be the solution of equation* (74)[9] *in the region* Ω *belonging to* $\overset{o}{D}$. *Then*

$$2\int_{\Omega_z} \left|\frac{\partial\omega}{\partial x_m}\right|^2 dx - \int_{\Gamma_z} (x_m+\tau)\left|\frac{\partial w}{\partial y}\right|^2 \cos(\nu,x_m)d\Gamma = \Psi(z), \qquad (75)$$

where τ *is a constant,* ν *is the outer normal, and*

$$\Psi_w(z) = (z+\tau)\int_{\eta_z}\left[\lambda|w|^2 + \left|\frac{\partial w}{\partial x_m}\right|^2 - \sum_{k=1}^{m-1}\left|\frac{\partial w}{\partial x_k}\right|^2\right]d\tilde{x} + \int_{\eta_z} w\,\frac{\partial \bar{w}}{\partial x_m}\,d\tilde{x}.$$

Proof: It follows from lemma 8 that the function w and its partial derivatives of first order are continuous up to the boundary Γ, and also that

$$w\Big|_{\Gamma} = 0. \tag{76}$$

It is not hard to verify that

$$\int_{\Omega_z}(x_m+\tau)\left[\frac{\partial w}{\partial x_m}\Delta\bar{w} + \frac{\partial \bar{w}}{\partial x_m}\Delta w\right]dx = \int_{\Omega_z}\left[\sum_{k=1}^{m-1}\left|\frac{\partial w}{\partial x_k}\right|^2 - \left|\frac{\partial w}{\partial x_m}\right|^2\right]dx$$

$$+\int_{\Gamma_z'}(x_m+\tau)\left[\frac{\partial w}{\partial x_m}\frac{\partial \bar{w}}{\partial \nu} + \frac{\partial \bar{w}}{\partial x_m}\frac{\partial w}{\partial \nu} - \sum_{k=1}^{m}\left|\frac{\partial w}{\partial x_k}\right|^2\cos(\nu,x_m)\right]d\,\Gamma',$$

where $\Gamma_z' = \Gamma_z + \eta_z$. It follows from (76) that

$$\frac{\partial w}{\partial x_k} = \frac{\partial w}{\partial \nu}\cos(\nu,x_k).$$

The last three equations, together with (74), now imply equation (75).

We remark that the identity (75) is analogous to an identity of Rellich [21].

We shall now impose several additional restrictions on the coefficients of our equation and on the region Ω. We will assume that $a_{k\ell} = \delta_{k\ell}$ everywhere in Ω, and that $g = -\Delta$. In addition, we shall assume that the cylindrical tube of Ω satisfies the following condition:

1) For any z the surface Γ_z is a Lyapunov surface.
2) Every straight line parallel to the x_m axis having a point in common with η intersects the surface Γ in one and only one point.
3) If the point $(x_1, \ldots, x_{m-1}, x_m)$ belongs to Ω, then the point $(x_1, \ldots, x_{m-1}, 0)$ belongs to η.

We shall call such a region a rectilinear cylindrical tube. In this special case it is possible to obtain stronger results than those proven above.

First of all, as shown by Rellich [21], the operator G does not have discrete eigenvalues. However Rellich's proof contains an error in connection with the fact that the eigenfunctions are in advance assumed to be analytic up to the boundary of Γ, and also the smoothness condition imposed on Γ are generally not formulated. We therefore present a corrected proof of that assertion.

THEOREM 10. *Let* Ω *be a rectilinear cylindrical tube and* $g = -\Delta$. *Then the operator* G *does not have discrete eigenvalues.*

Proof: Let us assume that λ is a discrete eigenvalue and $u(x)$ is the corresponding eigenfunction. Let the number τ be such that $x_m > -\tau$ in Ω. From lemma 9 and the inequality $\cos(\nu, x_m) \leq 0$, which is valid due to the conditions imposed on the boundary Γ, we have

$$2\int_{\Omega_z} \left|\frac{\partial u}{\partial x_m}\right|^2 dx \leq \Psi_u(z). \tag{77}$$

In view of the fact that $u \varepsilon W_2^{(1)}(\Omega)$, there exists a sequence $z_n \to \infty$ such that $\Psi(z_n) \to 0$. By the same token, it follows from (77) that $\frac{\partial u}{\partial x_m} = 0$ in Ω, which implies that $u = 0$ in Ω.

Q.E.D.

THEOREM 11. Let the conditions of theorem 10 be fulfilled. Then the assertion of theorem 7 holds for any real λ.

Proof: For a number λ which does not coincide with any μ_k, the assertion of the theorem follows immediately from theorems 7 and 10. Now let $\lambda = \mu_k$. Let u be the solution of the equation $\overline{G}u = \lambda u$, satisfying the conditions: $\frac{\partial u}{\partial x_m} \varepsilon L_2(\eta_z)$, (62), (63). With the aid of the argument employed in the proof of theorem 7, we obtain that the developments of the functions u and $\frac{\partial u}{\partial x_m}$ in the cross-section η_z in terms of the functions ϕ_k have the form:

$$u = a_p \phi_p + \sum_{k=p+1}^{\infty} a_k e^{-\sqrt{u_k-\lambda}\, z} \phi_k,$$

$$\frac{\partial u}{\partial x_m} = - \sum_{k=p+1}^{\infty} a_k \sqrt{\mu_k-\lambda}\, e^{-\sqrt{\mu_k-\lambda}\, z} \phi_k,$$

whence

$$\Psi_u(z) = \sum_{k=p+1}^{\infty} \sqrt{\mu_k-\lambda}\, |a_k|^2 e^{-2\sqrt{\mu_k-\lambda}\, z}.$$

It follows from lemma 9, as above, that inequality (77) is valid for the function u. Letting z tend to infinity, we obtain: $\frac{\partial u}{\partial x_m} = 0$ in Ω, which implies that $u = 0$ in Ω.

Q.E.D.

With the aid of theorem 11, the proofs of theorems 8 and 9 easily extend to the case of arbitrary λ.

THEOREM 12. *Let* Ω *be a rectilinear cylindrical tube, and* $g = -\Delta$. *Then the assertion of theorem* 8 *is valid for any real* λ. *Moreover, for any finite interval* $0 \leq \lambda \leq b$, *the assertion of theorem* 9 *holds, where* $\delta = 2$ *if the interval* $a \leq \lambda \leq b$ *does not contain any of the points* μ_k, *and where* $\delta = 1$ *if this rectangle contains even one such point*.

In the case $m = 3$ theorem 12 implies that for any interval $a \leq \lambda \leq b$ and for any finite interior subregion ω, there exists a constant c_{52} such that

$$|u_{\pm}(x,\lambda_2) - u_{\pm}(x,\lambda_1) \leq c_{52}|\lambda_2-\lambda_1|^{1/2},$$

where $a \leq \lambda_1 \leq \lambda_2 \leq b$, $x \in \omega$. This inequality implies, just as in §3, that in the case $m = 3$, $g = -\Delta$, and Ω is a rectilinear cylindrical tube, the assertion of theorem 6 is valid.

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FOOTNOTES

1. In equality (4) and what follows, c_k denotes a positive constant.

2. Condition (5), (6) were introduced by M. Sh. Birmanom in [5].

3. An analogous theorem in the case $g = -\Delta$, $\Omega = R_3$ was proved by Carleman [10].

4. Uniform convergence for $m > 3$ does not hold in general.

5. Here it is assumed, as in §2, that Ω is a region in three-dimensional space.

6. The integral is to be understood in the sense of a Cauchy principal value.

7. $\tilde{x}$ denotes a point in the space of variables $x_1, x_2, \ldots, x_{m-1}$.

8. Let $\mu_p < a < b < \mu_{p+1}$. Then for $k \leq p$, $\sqrt{\mu_k - \lambda} = i\sqrt{\lambda - \mu_k}$, where $\lambda \varepsilon [a,b]$.

9. The considerations of the paper are still true under the restriction $\alpha > 0$ only. One has to use Kato's [22] result about the growth properties of the solutions of reduced wave equation in place of the proof of theorem 3.

10. This restriction is not necessary, see Ikebe [23].

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